CO J = 1-0 is the most common tracer used to map molecular clouds. In this problem you will explore how we determine the optical depth and excitation temperature of optically thick CO 1-0 emission.

(a) Radio observations use position switching to subtract the contribution from the atmosphere. This also subtracts any other background such as the CMB. In an earlier homework problem, you derived the equation for background subtracted intensity: \( \Delta I_v = [B_v(T_{ex}) - B_v(T_{bg})](1 - e^{-\tau_v}) \). If we multiply \( \Delta I_v \) by \( c^2/2k\nu^2 \), the left side becomes the equivalent Rayleigh-Jeans radiation temperature \( T_R \). Show that the right side is the same equation but with each \( B_v(T) \) replaced with \( J_v(T) \) - the Planck function with units of temperature (K).

Figure 1: \(^{12}\)CO 1-0 integrated intensity of molecular clouds in the direction of Orion and Monoceros.
(b) One way to determine the optical depth is to observe two isotopologues (fancy word for isotopes in molecules) in the same transition. This is commonly done by observing $^{12}\text{CO} \text{ 1-0}$ and $^{13}\text{CO} \text{ 1-0}$. If the abundance ratio of $^{12}\text{C}/^{13}\text{C} \sim 50$ and there is no change in this ratio between atoms and molecules, then $\tau(^{12}\text{CO} \text{ 1-0})/\tau(^{13}\text{CO} \text{ 1-0}) \sim 50$. Using this fact and the equation you derived in part (a) to write down a non-linear equation for optical depth in $^{12}\text{CO} \text{ 1-0}$ (Hint take a ratio of the radiative transfer equations and eliminate $\tau(^{13}\text{CO} \text{ 1-0})$). Assume $^{12}\text{CO}$ and $^{13}\text{CO}$ transitions are at the same frequency (approximately true!), have the same $T_{\text{ex}}$, and $^{13}\text{CO} \text{ 1-0}$ emission is optically thin.

(c) If both $^{12}\text{CO}$ and $^{13}\text{CO} \text{ 1-0}$ were optically thin, what would be the ratio in observed $T_R$? This is an easy way to observationally check if emission is optically thin using isotopologues.

(d) If observations of a molecular cloud have $^{12}\text{CO} \text{ 1-0} T_R = 15 \text{ K}$ and $^{13}\text{CO} \text{ 1-0} T_R = 5 \text{ K}$ at the line peak, calculate the peak $\tau(^{12}\text{CO} \text{ 1-0})$.

(e) Calculate $T_{\text{ex}}$ for the $^{12}\text{CO} \text{ 1-0}$ transition at the line peak.

Figure 2: Spectra of isotopologues of $\text{CO J = 1-0}$ toward a molecular cloud.