19. Let's calculate the long wavelength “spectral index” of dust emission. The spectral index, $\alpha$, is defined as the power-law slope of the observed flux density over some range of wavelengths: $F_\nu \sim \nu^\alpha$. Consider an **optically thin** source of dust emission with emission well described by a source function $B_\nu(T_d)$ over a solid angle $\Omega$, and dust mass opacity parametrized by $\kappa(\nu) = \kappa_0 \left(\frac{\nu}{\nu_0}\right)^\beta$ (cm$^2$/g of dust) at long wavelengths. Assume that the source is small and $F_\nu = I_\nu \Omega$ is a good approximation.

(a) How does the dust optical depth $\tau_\nu$ depend on frequency?

(b) In the Rayleigh-Jeans limit at long wavelength what is the spectral index $\alpha$ for (1) blackbody emission and (2) for optically thin dust emission with a dust opacity index of $\beta$? Dust emission has a “steeper spectral index” than blackbody emission (see Figure).
20. Consider an optically thin dusty disk surrounding a young protostar with luminosity $L_{\text{star}}$. In this problem, you will derive how the temperature of dust grains varies with distance, $r$, from the star.

(a) Assume the dust opacity at long wavelengths is given by a power-law $Q_{\text{abs}} \sim \nu^\beta$. Derive the dependence of $T_d$ with $r$ and $\beta$. I just want you to write down how $T_d$ is proportional to $r$ and $\beta$ (don’t have to write out all the constants, etc.).

(b) In typical ISM dust, $\beta \sim 2$, but in young protostellar disks where dust grain start to grow into mm sized grains (“dust coagulation”), we find that $\beta \sim 1$ (or smaller). Large (km) planetesimals would radiate closer to a blackbody and would have $\beta \sim 0$. Thus, there is significant evolution in the opacity ($Q_{\text{abs}}$ or $\kappa_{\nu}$) of dust grains in planet-forming disks (see Figure below for a calculation of how $\beta$ at long wavelengths varies with size of grains). Calculate the power-law indicies of $T_d$ with $r$ for $\beta = 2$, 1, and 0 (simplify any fractions).

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