Einstein Coefficients

Let's analyze a 2 level system in the limit of \( n_c \to 0 \)

\[
n_1 B_{12} u_\nu = n_2 A_{21} + n_2 B_{21} u_\nu
\]

Solve for \( u_\nu \) and divide both sides by \( n_2 B_{21} \)

\[
\frac{n_1 B_{12}}{n_2 B_{21}} u_\nu - u_\nu = \frac{A_{21}}{B_{21}}
\]

\[
u_\nu = \frac{A_{21}}{B_{21}} \cdot \frac{1}{\frac{B_{12}}{B_{21}} \frac{n_1}{n_2} - 1}
\]

Boltzmann: \( \frac{n_1}{n_2} = \frac{g_1}{g_2} e^{\frac{\Delta E_{\nu \nu}}{kT}} \) because \( \Delta E_{\nu \nu} = \hbar \nu \)

\[
u_\nu = \frac{A_{21}}{B_{21}} \cdot \frac{1}{\frac{B_{12}}{B_{21}} \frac{g_1}{g_2} e^{\frac{\hbar \nu}{kT}} - 1}
\]

But \( \nu_\nu = \frac{8 \pi \hbar \nu^3}{c^3} \) for a blackbody

\[
\Rightarrow A_{21} = \frac{8 \pi \hbar \nu^3}{c^3} B_{21} \quad \text{and} \quad g_1 B_{12} = g_2 B_{21}
\]

Can convert Einstein \( B \)s to \( A \)s and vice versa.

Note: we derived this in a particular limit assuming \( u_\nu \) was a blackbody — but these are constants so are always true.
The exact equation depends on the "multipole" of radiation. Radiation is generated by acceleration of charge and the "Strongest" multipole is **Electric Dipole Radiation**.

From Quantum point-of-view \( \langle P \rangle = h \nu \cdot A_{ue} \) erg s\(^{-1}\) erg \( \cdot \) s\(^{-1}\) \( \checkmark \)

Let's relate this average power radiated to the average power from an oscillating electric dipole. \( d = e \cdot \chi(t) \)

\[ \chi(t) = \chi_0 \cos\omega t \quad \text{where} \quad \omega = 2\pi \nu \]

\[ \ddot{\chi}(t) = \chi_0 \omega^2 \cos\omega t \quad \text{is acceleration of the charge} \]

The power radiated into 4\(\pi\) ster from accelerating charge is:

\[ P(t) = \frac{2}{3} \frac{e^2 \omega^2 \chi(t)}{c^3} \quad \int_0^{2\pi/\omega} \cos^2 \omega t \, dt = \frac{\pi}{\omega} \]

\[ \langle P \rangle = \frac{2}{3c^3} \cdot \frac{e^2 \chi_0^2 (2\pi\nu)}{2\pi/\omega} \cdot \int_0^{2\pi/\omega} \cos^2 \omega t \, dt = \frac{2\pi \nu}{\omega} \]

Define the average electric dipole as \( \overline{Me} = \frac{e\chi_0}{\nu} \Rightarrow Me = \frac{e^2 \chi_0^2}{4} \)

\[ \langle P \rangle = \frac{64\pi \nu^4 \chi_0^2}{3c^3} \overline{Me}^2 = h \nu \cdot A_{ue} \]

\[ \Rightarrow A_{ue} = \frac{64\pi \nu^4 \chi_0^2}{3hc^3} \left| \mu_{ue} \right|^2 \]

**Electric Dipole \( \sim \nu^3 \)** (also magnetic dipole)

**NOTE**: Turns out that **Electric Quadrupole \( \sim \nu^5 \)** (also magnetic quadrupole); etc.