Optical Depth $\tau_\nu$ & Absorption Coefficient $\alpha_\nu$ :

$$\tau_\nu = \int_0^L \alpha_\nu ds$$

$\alpha_\nu$ = # of photon absorptions per cm$^{-1}$

So really $\tau_\nu$ is given by

$$\tau_\nu = \int_0^L n \sigma_\nu ds = \sigma_\nu \int_0^L n ds$$

If $\sigma_\nu$ independent of $s$

This is the Column Density $N = \int_0^L n ds$ cm$^{-2}$

Column Density is a measure of how much "stuff" you have along a line of sight (in a particular direction).

$$\tau_\nu = \sigma_\nu \cdot N$$

Optical depth = cross-section of absorbers $\times$ Column Density of absorber
Practical Example:

Assume you have 2 clouds with absorbers with cross-sections $\theta_v$ (same in both clouds) and $n$ same in both clouds.

Cloud 1

\[ N_1 = \int_0^L n ds = n \cdot L \]

Cloud 2

\[ N_2 = \int_0^{2L} n ds = 2nL \]

\[ \frac{\tau_v^{(\text{cloud 2})}}{\tau_v^{(\text{cloud 1})}} = \frac{2\theta_v \cdot n \cdot L}{\theta_v \cdot n \cdot L} = 2 \]

Optical depth scales linearly with how much "stuff" you have along a line of sight.
Let's calculate expectation value of $\mathcal{T}_\nu$

$$
\langle \mathcal{T}_\nu \rangle = \int_0^\infty \mathcal{T}_\nu \cdot P(\mathcal{T}_\nu) \ d\mathcal{T}_\nu
$$

**Probability of absorption in $[\mathcal{T}_\nu, \mathcal{T}_\nu + d\mathcal{T}_\nu]$

**Probability of non-absorption in $[0, \mathcal{T}_\nu]$
and absorption within $d\mathcal{T}_\nu$

\[
\text{Prob. of photon absorbed in } [0, \mathcal{T}_\nu] \quad \Rightarrow \quad P(0, \mathcal{T}_\nu) = \frac{\Delta I(\mathcal{T}_\nu)}{I_0} = \frac{I_0 - I(\mathcal{T}_\nu)}{I_0} = 1 - \frac{I(\mathcal{T}_\nu)}{I_0}
\]

\[
\text{Prob. of non-absorption in } [0, \mathcal{T}_\nu] \quad \Rightarrow \quad 1 - P(0, \mathcal{T}_\nu) = \frac{I(\mathcal{T}_\nu)}{I_0} = e^{-\mathcal{T}_\nu}
\]

**Prob. of absorption in $d\mathcal{T}_\nu$

\[
P(\mathcal{T}_\nu, \mathcal{T}_\nu + d\mathcal{T}_\nu) = \frac{dI(\mathcal{T}_\nu)}{I(\mathcal{T}_\nu)} = d\mathcal{T}_\nu
\]

Total Prob. is product of Prob of non-absorb $\times$ Prob of absorb in $d\mathcal{T}_\nu$

\[
\langle \mathcal{T}_\nu \rangle = \int_0^\infty \mathcal{T}_\nu \ e^{-\mathcal{T}_\nu} \ d\mathcal{T}_\nu
\]

\[
= - \left( \frac{1}{e^\infty} + \frac{e^\infty}{\infty} \right) + (1 + 0)e^0
\]

\[
= 1
\]

When $\tau = 1 \quad \mathcal{T}_\nu = \int_0^\mathcal{L} \alpha' \ ds = \alpha' \cdot L
$

\[
\Rightarrow \quad 1 = \alpha' \cdot L \quad \Rightarrow \quad L = \frac{1}{\alpha'} = \frac{1}{n\theta} \quad \text{Mean free path length}
\]