Scale Factor - Cosmology

If the expansion of spacetime is homogeneous & isotropic (Cosmological Principle), then the distance between any 2 points is:

$$r(t) = a(t)\, r_0$$

← subscript means present epoch = today

↑ Scale factor (dimensionless)

If spacetime is expanding or contracting:

$$V(t) = \frac{dr(t)}{dt} = \frac{da(t)}{dt} \cdot r_0 = \dot{a}\, r_0$$

← dot notation means time derivative

but $r_0 = \frac{r(t)}{a(t)} \Rightarrow V(t) = \frac{\dot{a}(t)}{a(t)}\, r(t)$ This is the Hubble Law!

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}$$ is the Hubble "Constant" (really a function of time) ← Really called Hubble Parameter.

H_0 = Hubble constant today

We can relate the scale factor back to redshift.

$$\frac{\lambda_{obs}}{\lambda_{emit}} = \frac{a(t_0)}{a(t_{emit})} = \frac{a_0 = 1}{a(t_{emit})}$$

$$1 + z = \frac{\lambda_{obs}}{\lambda_{emit}} = \frac{1}{a(t_{emit})}$$

$\Rightarrow$ when $z = 1$ then $a = \frac{1}{2} \Rightarrow$ Universe half present "size"

How does scale factor change? We need a solution to the Einstein Field Equations (General Relativity) in the limit of a homogeneous, isotropic Universe:

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = \frac{8\pi G}{3}\left[\rho_{m,0}\, a^{-3} + \rho_{r,0}\, a^{-4} + \rho_{\Lambda,0}\right] - \frac{Kc^2}{a^2}$$

Friedmann Equation

mass density at present

radiation density at present

Cosmological Constant

Curvature of spacetime. CMB results $K \approx 0$ (flat geometry)

We can rewrite this equation in terms of

$$\Omega_s \equiv \rho/\rho_{crit} \quad \text{where} \quad \rho_{crit} = \frac{3H^2(t)}{8\pi G}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = H_0^2\left[\Omega_{m,0}\, a^{-3} + \Omega_{r,0}\, a^{-4} + \Omega_{\Lambda,0}\right]$$

Hubble "constant" at present epoch

Note for flat spacetime

$$\Omega_m + \Omega_r + \Omega_\Lambda = \Omega_{tot} = 1$$

Where does dependence on a come from in each term?

$\rho_{m,0}\, a^{-3}$: $\rho_m(t) = \dfrac{M}{V(t)} \sim \dfrac{M}{a^3(t)} \sim a^{-3}$

$\rho_{r,0}\, a^{-4}$: Wiens Law $\lambda_{max} \cdot T_{BB} = \text{CONST}$

Since $\lambda \sim a(t)$ (redshift)

then $T_{BB} \sim \dfrac{1}{a}$

Energy density of BB $u_{BB} = \rho_r c^2 \sim T^4 \Rightarrow \rho_r \sim \dfrac{1}{a^4} \sim a^{-4}$

$\rho_{\Lambda,0}$: Λ is cosmological constant (really constants of integration from Einstein Field Equations)

$\Rightarrow$ no dependence on a

Let's solve the Friedmann Equation in the limits that different energy densities dominate: (3)

(1.) Ω_r Dominates – Radiation Dominated

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2 \Omega_{r,0}\, a^{-4}$$

$$a^{\cancel{2}} \frac{\left(\frac{da}{dt}\right)}{\cancel{a}} = H_0 \Omega_{r,0}^{1/2}$$

$$\int a\, da = H_0 \Omega_{r,0}^{1/2} \int dt$$

$$\frac{1}{2} a^2 + \text{CONSTS} = H_0 \Omega_{r,0}^{1/2} \cdot t + \text{MORE CONST}$$

$$\Rightarrow \quad a^2 \sim t$$

$$a \sim t^{1/2}$$

$$\dot a \sim t^{-1/2}$$

$$\ddot a \sim -t^{-3/2} < 0 \Rightarrow$$

IN radiation dominated Universe, expansion is Slowing down.

(2.) Ω_m Dominates – Matter Dominated

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2 \Omega_{m,0}\, a^{-3}$$

$$\int a^{1/2} da = H_0 \Omega_{m,0}^{1/2} \int dt$$

$$a^{3/2} \sim t$$

$$a \sim t^{2/3}$$

← grows faster than in radiation dominated case ⇒ at some point matter will dominate over radiation!

$$\dot a \sim t^{-1/3}$$

$$\ddot a \sim -t^{-4/3} < 0 \Rightarrow$$

IN Matter dominated Universe expansion is also Slowing down

(3) Ω_Λ Dominates - Dark Energy Dominated

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\,\Omega_{\Lambda,0}$$

$$\int \frac{da}{a} = H_0 \Omega_{\Lambda,0}^{1/2} \int dt$$

$$\ln a \sim H_0 \Omega_{\Lambda_0}^{1/2} \cdot t$$

$$a \sim e^{H_0 \Omega_{\Lambda,0}^{1/2} t}$$

$$\dot a \sim + e^{H_0 \Omega_{\Lambda,0}^{1/2} \cdot t}$$

$$\ddot a \sim + e^{H_0 \Omega_{\Lambda,0}^{1/2} \cdot t} > 0 \Rightarrow$$

In Λ dominated Universe, the expansion is accelerating

Note - In General you can solve the Friedmann Equation numerically to find $a(t)$.

Allows you to calculate things like

age of Universe at z $\quad t(z) = \int_0^{a(z)} \frac{da}{\dot a(z)}$ etc.

too advanced for ASTR 250 but something to look forward to!