Virial Theorem

Derivation

One of the most important theorems in astrophysics imagine a system of $N$ bound particles with masses, $m_i$, at a distance $\vec{r}_i$ from the center of mass.

Each particle has momentum $\vec{p}_i = m_i \vec{v}_i$

Define "the Virial"

$$A = \sum_{i=1}^{N} m_i \vec{v}_i \cdot \vec{r}_i = \sum_{i=1}^{N} \vec{p}_i \cdot \vec{r}_i$$

Now take $\frac{d}{dt}$ of $A$:

$$\frac{dA}{dt} = \sum_i \frac{d\vec{p}_i}{dt} \cdot \vec{r}_i + \vec{p}_i \cdot \frac{d\vec{r}_i}{dt}$$

$$= \sum_i m_i \frac{d\vec{v}_i}{dt} \cdot \vec{r}_i + m\vec{v}_i \cdot \vec{v}_i$$

Note: $\vec{v}_i \cdot \vec{v}_i = v_i^2$

$$= \sum_i m_i \vec{a}_i \cdot \vec{r}_i + m\vec{v}_i^2$$

Note: $\vec{F}_i = m\vec{a}_i$

$$= \sum_i \vec{F}_i \cdot \vec{r}_i + 2T$$

where $T = \text{kinetic energy of all particles}$

$$T = \frac{1}{2} m v^2$$
Now take the time average of \( \frac{dA}{dt} \):

\[
\langle \frac{dA}{dt} \rangle = \frac{1}{t} \int_0^t \frac{dA}{dt} \, dt
\]

\[
= \langle \sum_i \vec{F}_i \cdot \vec{r}_i \rangle + \langle 2T \rangle
\]

If the system of particles is bound, then \( \vec{r}_i \) and \( \vec{p}_i \) remain finite. \( \implies \sum \vec{r}_i \vec{r}_i \) is finite \( \implies \) so is its time derivative. The system of particles is "virialized" if after a long enough time, \( \langle \frac{dA}{dt} \rangle \rightarrow 0 \).

Virialized \( \implies \langle 2T \rangle = -\langle \sum_i \vec{F}_i \cdot \vec{r}_i \rangle \)

we can relate this to the potential energy of the system.

In general \( \vec{F}_i = -\nabla U(r_i) = -\frac{dU}{dr_i} \) for a purely radial force.

E.g. gravity \( F = -\frac{6Mm}{r^2} \) and \( U = \frac{6Mm}{r} \) = potential energy.

If \( U \propto r^n \) then \( F \propto r^{n-1} \). For gravity where \( F \propto r^{-2} \)

\[
\frac{dU}{dr} \propto nr^{n-1}
\]

so \( \frac{dU}{dr} \propto n \cdot r^n \cdot 2U \).
So the term
\[
- \sum_i \mathbf{F}_i \cdot \mathbf{r}_i = + \sum_i \frac{dU}{dr_i} r_i
\]
\[
= + \sum_i n U(r_i)
\]
\[
= + n U
\]

Back to the virial theorem:
\[
\langle 2T \rangle = + n \langle U \rangle
\]

For gravity \( n = -1 \) so
\[
\langle T \rangle = - \frac{1}{2} \langle U \rangle
\]

The Virial Theorem applies when

1. Bound system of particles
2. Force holding system together has purely radial dependence (e.g. \( F_{\text{grav}} \propto r^{-2} \))
3. System is in "virial equilibrium" (or is "virialized") such that \( \langle \frac{dA}{dt} \rangle \to 0 \).

Another way of saying the above statement is that over a long period of time, the virial, \( A \), is constant.