Nucleosynthesis

Review: Fusion converts $^4_1H \rightarrow ^4_2He$

$$\Delta m = 4m(^1H) - m(^4He) = 0.0291 \text{ amu}$$

$$\frac{\Delta m}{4} = 0.7\% \text{ MH}$$

$$\Delta E = \Delta mc^2 \approx 27 \text{ MeV}$$

How does this conversion actually occur?

This is what we are going to figure out today.

Problem: $^4_1H \rightarrow ^4_2He$

Collision of 2 protons. Protons are + charged.
Implies we have to overcome electrostatic repulsion to combine them into a nucleus.

1st - What kind of $T$ can we expect in the Sun?

Let's estimate from the Virial Theorem!

$$\left\langle \mathbf{K.E.} \right\rangle = -\frac{1}{2} \left\langle U \right\rangle$$

For a "perfect" gas $<N_kT>$ number of gas particles

$$K.E. = \frac{3}{2} N k T_{vir}$$

$$U = -\frac{3}{5} \frac{GM^2}{R} \leftarrow \text{ we calculated this last time.}$$

So

$$\frac{3}{2} N k T_{vir} = -\frac{1}{2} \cdot -\frac{3}{5} \frac{GM^2}{R}$$
\[ \frac{3}{2} \pi k T_{\text{vir}} = \frac{3}{10} \frac{GM^2}{R} \]

\[ \uparrow \]

want to get rid of \( N \) in this equation, so let's substitute \( M^2 = M \cdot M = M \cdot (N \cdot \overline{m}) \)

\[ \overline{m} = \frac{m_p + m_e}{2} \approx \frac{m_p}{2} \]

so \[ \frac{3}{2} \pi k T_{\text{vir}} = \frac{1}{10} \frac{GM \overline{m} m_p}{R^2} \]

\[ T_{\text{vir}} = \frac{GM \overline{m} m_p}{10 \pi k R_0} \]

\[ = \frac{(6.67 \times 10^{-8}) \ (2 \times 10^{33} \text{ g}) \ (1.67 \times 10^{-24})}{10 \ (1.38 \times 10^{16} \text{ erg.K}^{-1}) \ (7 \times 10^{10} \text{ cm})} \]

\[ \approx 2 \times 10^6 \text{ K} \]

Very hot!

But is this hot enough for fusion?
Potential Energy Surface for Colliding Protons:

Coulomb Potential $U \sim \frac{1}{r}$ between two charged protons

$R_0 \approx 10^{-13}$ cm $\approx$ size of nucleus

Strong nuclear force is attractive for $r < R_0$!

$$U_{\text{coulomb}} = \frac{q^2}{r}$$

$q = \text{charge of proton} = 4.8 \times 10^{-10}$ esu (cgs units)

$= 1.6 \times 10^{-19}$ C (MKS units)

So the potential barrier that must be overcome is:

$$U_{\text{coulomb}} = \frac{q^2}{R_0} = \frac{(4.8 \times 10^{-10} \text{ esu})^2}{10^{-13} \text{ cm}} \approx 2 \times 10^{-6} \text{ erg}$$

OK, so how much kinetic energy is there if $T \approx 10^7$ K gas?

$$E_{\text{kinetic}} = \frac{3}{2} k T = \frac{3}{2} (1.38 \times 10^{-16} \text{ erg} \cdot \text{K}) \cdot 10^7 \text{ K} \approx 2 \times 10^{-9} \text{ erg}$$

$$\Rightarrow E_{\text{kinetic}} \ll E_{\text{coulomb}} \text{ by a factor of 1000}$$
In reality, that is the average kinetic energy, but some particles have higher kinetic energies. A gas follows a Maxwellian Velocity Distribution:

\[
N(v) \sim e^{-\frac{mv^2}{2kT} - \frac{E_{\text{kin}}}{kT}}
\]

exponential "tail" of proton energies.

So \( N(v > \text{Coulomb barrier}) \sim e^{-\frac{E_{\text{kin}}}{E_{\text{Coulomb}}} - 1000} \sim 10^{-430} \) (yikes!)

Not even \( \frac{1}{20} \) proton in the Sun \( @ T \sim 10^7 K \) has enough kinetic energy to overcome the electrostatic repulsion between protons \( \frac{1}{20} \).

Classically, fusion should not occur.

The answer: Quantum Mechanics to the rescue!

"Quantum Tunneling"
According to Quantum Mechanics, particles are described by a wavefunction, \( \psi \), and the probability of finding a particle at a position is space \( \sim |\psi|^2 \).

One property of wavefunctions is that they exist everywhere in space. So, there is some non-zero probability that the proton will "tunnel" through the Coulomb barrier.

\[ \psi = \text{proton wavefunction} \]

In Quantum Mechanics, you can show that wavefunctions exponentially decay across a potential barrier.

The calculation of this probability is too advanced for this class — but it is high enough that it does occur!

The reaction is:

\[ ^1H + ^1H \rightarrow ^2D + e^+ + \nu_e \]

Deuterium is just an isotope of hydrogen w/ 1 proton + 1 neutron.
Wait a minute — we have 2 protons on the left side, but \(^2\text{D}\) has 1 proton & 1 neutron. How did the proton convert into a neutron?²

\[
\text{β-decay} \quad p^+ \rightarrow n + e^+ + \gamma_e
\]

This reaction is controlled by the nuclear "weak" force. It is very slow.

The timescale for a proton to (1) tunnel through the Coulomb barrier and (2) undergo β decay is \(\approx 10^{10}\) years in the Sun! Very Slow.

The p-p chain is how the Sun gets 91% of its energy:

\[
\begin{align*}
\text{p-p I} & : & \quad ^1\text{H} + ^1\text{H} & \rightarrow ^2\text{D} + e^+ + \gamma_e & \tau & \sim 10^{10} \text{ years} \\
& & ^2\text{D} + ^1\text{H} & \rightarrow ^3\text{He} + \gamma_{\text{gamma photon}} & \tau & \sim 6 \text{ secs}! \\
& & ^3\text{He} + ^3\text{He} & \rightarrow ^4\text{He} + f \text{He} + \text{H} + \text{H} & \tau & \sim 10^{6} \text{ years}
\end{align*}
\]

\(\Delta E_{\text{tot}} \approx 26.2 \text{ MeV}\) (excludes neutrino energies since they don't interact strongly w/ matter in the Sun and escape easily)

Number of reactions in the Sun:

\[
N_{\text{reactions}} = \frac{L_0}{\Delta E_{\text{reaction}}} = \frac{3.9 \times 10^{33} \text{ erg s}^{-1}}{26.2 \times 10^6 \text{ erg s}^{-1} \cdot 1.602 \times 10^{-12} \text{ erg/ev}} \approx 10^{38} \text{ s}^{-1}
\]

So, even though 1 reaction is rare, there are \(10^{38}\) protons in the Sun — so a lot of reactions are still occurring!