Integrating the Planck Integral

Multiply by e^{-x} / e^{-x}

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \int_0^\infty \frac{x^3 e^{-x}}{1 - e^{-x}} dx = \int_0^\infty x^3 \sum_{n=1}^{\infty} e^{-nx} dx$$

Turn into the series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Let's look at each term in the sum (we can do this because each term in the sum is a convergent integral)

Substitute u = nx

$$\int_0^\infty x^3 e^{-nx} dx = \frac{1}{n^4} \int_0^\infty u^3 e^{-u} du.$$

This integral is a Gamma Function:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx. \qquad \Gamma(n) = (n-1)!$$

$\Gamma(4) = 3! = 6$

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \sum_{n=1}^{\infty} \frac{6}{n^4}.$$

This sum is a Reimann Zeta Function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$\zeta(4) = \pi^4/90$

This result can be generalized to yield

$$\int_0^\infty \frac{x^\alpha}{e^x - 1} dx = \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+1)}{n^\alpha} = \Gamma(\alpha+1)\zeta(\alpha+1).$$

Source: Jeremy Heyl. "A.1 The Integral of x^3/(e^x - 1)." *Theoretical Astrophysics* Boundless, 26 May. 2016. Retrieved 22 Jan. 2017 from https://www.boundless.com/users/235417/textbooks/theoretical-astrophysics/appendix-a-15/mathematical-appendix-99/a-1-the-integral-of-x-3-e-x-1-237-16449/