very simple way. If we denote the state $|\frac{1}{2}, +\frac{1}{2}\rangle$ by α and the state $|\frac{1}{2}, -\frac{1}{2}\rangle$ by β, then the general expressions given earlier become

$$s_z\alpha = +\tfrac{1}{2}\hbar\alpha \qquad s_z\beta = -\tfrac{1}{2}\hbar\beta \qquad s^2\alpha = \tfrac{3}{4}\hbar^2\alpha \qquad s^2\beta = \tfrac{3}{4}\hbar^2\beta \tag{35}$$

and the effects of the shift operators are

$$s_+\alpha = 0 \qquad s_+\beta = \hbar\alpha \qquad s_-\alpha = \hbar\beta \qquad s_-\beta = 0 \tag{36}$$

It follows that the only nonzero matrix elements of the shift operators are

$$\langle\alpha|s_+|\beta\rangle = \hbar \qquad \langle\beta|s_-|\alpha\rangle = \hbar \tag{37}$$

The angular momenta of composite systems

We now consider a system in which there are two sources of angular momentum, which we denote $\mathbf{j}_1$ and $\mathbf{j}_2$. The system might be a single particle that possesses both spin and orbital angular momentum, or it might consist of two particles with spin or orbital momentum. The question we investigate here is what the commutation rules imply for the total angular momentum $\mathbf{j}$ of the system.

4.9 The specification of coupled states

The state of particle 1 is fully specified by reporting the quantum numbers j_1 and m_{j1}, and the same is true of particle 2 in terms of its quantum numbers j_2 and m_{j2}. If we are to be able to specify the overall state as $|j_1m_{j1};j_2m_{j2}\rangle$, we need to know whether all the corresponding operators commute with one another. In fact, operators for independent sources of angular momentum do commute with one another, and we can write

$$[j_{1q}, j_{2q'}] = 0 \tag{38}$$

for all components q and q'. One way to see that this is so is to note that in the position representation the operators are expressed in terms of the coordinates and derivatives of each particle separately, and the derivatives for one particle treat the coordinates of the other particle as constants. *Operators that refer to independent components of a system always commute with one another.* Because the operators j_1^2 and j_2^2 are defined in terms of their components, which commute, so too do these two operators. Hence, all four operators j_1^2, j_{1z}, j_2^2, and j_{2z} commute with one another, and it is permissible to express the state as $|j_1m_{j1};j_2m_{j2}\rangle$.

We now explore whether the **total angular momentum**, $\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2$, can also be specified. First, we investigate whether $\mathbf{j}$ is indeed an angular momentum. To do so, we evaluate the commutators of its components, such as

$$\begin{aligned}[j_x, j_y] &= [j_{1x} + j_{2x}, j_{1y} + j_{2y}] \\ &= [j_{1x}, j_{1y}] + [j_{2x}, j_{2y}] + [j_{1x}, j_{2y}] + [j_{2x}, j_{1y}] \\ &= i\hbar j_{1z} + i\hbar j_{2z} + 0 + 0 = i\hbar j_z\end{aligned} \tag{39}$$

This commutation relation, and the other two that can be derived from it by cyclic permutation of the coordinate labels, is characteristic of angular

momentum, so $\mathbf{j}$ is an angular momentum ($\mathbf{j}_1 - \mathbf{j}_2$, on the other hand, is not). Because $\mathbf{j}$ is an angular momentum, we can conclude without further work that its magnitude is $\{j(j+1)\}^{1/2}\hbar$ with j integral or half-integral, and its z-component has the values $m_j\hbar$ with $m_j = j, j-1, \ldots, -j$.

We now need to work towards discovering which values of j can exist in the system. The initial question is whether we can actually specify j if j_1 and j_2 have been specified. Because j_1^2 commutes with all its components, and j_2^2 does likewise, and because j^2 can be expressed in terms of those same components it follows that

$$[j^2, j_1^2] = [j^2, j_2^2] = 0 \tag{40}$$

Therefore, we can conclude that the eigenvalues of j_1^2, j_2^2, and j^2 can be specified simultaneously. For instance, a p-electron (for which $l = 1$ and $s = \frac{1}{2}$) can be regarded as having a well-defined total angular momentum with a magnitude given by some value of j (the actual permitted values of which we have yet to find).

Because j^2 commutes with its own components, in particular it commutes with $j_z = j_{1z} + j_{2z}$. Therefore, we know that we can specify the value of m_j as well as j. At this point, we have established that a state of coupled angular momentum can be denoted $|j_1j_2;jm_j\rangle$. Note, however, that we have not yet established that it can be specified more fully as $|j_1m_{j1}j_2m_{j2};jm_j\rangle$ because we have not yet established whether j^2 commutes with j_{1z} and j_{2z}.

To explore this point we proceed as follows:

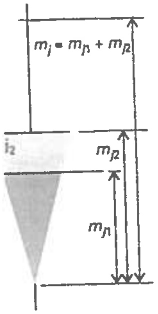
$$\begin{aligned}[j_{1z}, j^2] &= [j_{1z}, j_x^2] + [j_{1z}, j_y^2] + [j_{1z}, j_z^2] \\ &= [j_{1z}, (j_{1x} + j_{2x})^2] + [j_{1z}, (j_{1y} + j_{2y})^2] + [j_{1z}, (j_{1z} + j_{2z})^2] \\ &= [j_{1z}, j_{1x}^2 + 2j_{1x}j_{2x}] + [j_{1z}, j_{1y}^2 + 2j_{1y}j_{2y}] \\ &= [j_{1z}, j_{1x}^2 + j_{1y}^2] + 2[j_{1z}, j_{1x}]j_{2x} + 2[j_{1z}, j_{1y}]j_{2y} \\ &= [j_{1z}, j_1^2 - j_{1z}^2] + 2i\hbar j_{1y}j_{2x} - 2i\hbar j_{1x}j_{2y} \\ &= 2i\hbar(j_{1y}j_{2x} - j_{1x}j_{2y})\end{aligned} \tag{41}$$

The commutator is *not* zero, and so we *cannot* specify m_{j1} (or m_{j2}) if we specify the value of j.

It follows from this analysis that we have to make a choice when specifying the system. Either we use the **uncoupled picture** $|j_1m_{j1};j_2m_{j2}\rangle$, which leaves the total angular momentum unspecified and therefore, in effect, says nothing about the relative orientation of the two momenta, or we use the **coupled picture** $|j_1j_2;jm_j\rangle$ which leaves the individual components unspecified. At this stage, which choice we make is arbitrary. Later, when we consider the energy of interaction between different angular momenta we shall see that one picture is more natural than the other. At this stage, the two pictures are simply alternative ways of specifying a composite system.

4.10 The permitted values of the total angular momentum

If we decide to use the coupled picture, the question arises as to the permissible values of j and m_j. We know that the commutation relations permit j to have any positive integral or half-integral values, but we need to determine which o



4 A representation of the ment that $m_j = m_{j1} + m_{j2}$.

these many values actually occur for a given j_1 and j_2. For example, the total angular momentum of a p-electron ($l = 1$ and $s = \frac{1}{2}$) is unlikely to exceed $j = \frac{3}{2}$.

The allowed values of m_j follow immediately from the relation $j_z = j_{1z} + j_{2z}$, and are

$$m_j = m_{j1} + m_{j2} \tag{42}$$

That is, the total component of angular momentum about an axis is the sum of the components of the two contributing momenta (Fig. 4.4).

To determine the allowed values of j, we first note that the total number of states in the uncoupled picture is $(2j_1 + 1)(2j_2 + 1) = 4j_1j_2 + 2j_1 + 2j_2 + 1$. There is only one state in which both components have their maximum values, $m_{j1} = j_1$ and $m_{j2} = j_2$, and this state corresponds to $m_j = j_1 + j_2$. However, the maximum value of m_j is by definition j, so the maximum value of j is $j = j_1 + j_2$. There are $2j + 1 = 2j_1 + 2j_2 + 1$ states corresponding to this value of j, and so there are a further $4j_1j_2$ states to find.

Although the state with $m_j = j_1 + j_2$ can arise in only one way, the state with $m_j = j_1 + j_2 - 1$ can arise in two ways, from $m_{j1} = j_1 - 1$ and $m_{j2} = j_2$ and from $m_{j1} = j_1$ and $m_{j2} = j_2 - 1$. The state with $j = j_1 + j_2$ accounts for only one of these states (or for one of their two linear combinations), and so there must be another coupled state for which the maximum value of m_j is $m_j = j_1 + j_2 - 1$. This state corresponds to a state with $j = j_1 + j_2 - 1$. A system with this value of j accounts for a further $2j + 1 = 2j_1 + 2j_2 - 1$ states. The process can be continued by considering the next lower value of m_j, which is $m_j = j_1 + j_2 - 2$, and which can be produced in three ways. The two states with $j = j_1 + j_2$ and $j = j_1 + j_2 - 1$ account for two of them; the third (or the third linear combination) must arise from the state with $j = j_1 + j_2 - 2$. This argument can be continued, and all the states are accounted for by the time we have reached $j = |j_1 - j_2|$ (j is a positive number, hence the modulus signs). Therefore, the permitted states of angular momentum that can arise from a system composed of two sources of angular momentum are given by the **Clebsch–Gordan series**:

$$j = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2| \tag{43}$$

To verify that this series does indeed account for all $4j_1j_2 + 2j_1 + 2j_2 + 1$ states, we sum the number of states $(2j + 1)$ for each permitted value of j. For $j_1 \geq j_2$ the sum is

$$\sum_{j=j_1-j_2}^{j_1+j_2} (2j + 1) = 4j_1j_2 + 2j_1 + 2j_2 + 1 = (2j_1 + 1)(2j_2 + 1) \tag{44}$$

as required.[2]

[2] The sum of an arithmetical progression with n terms starting at a_1 and terminating at a_n is $\frac{1}{2}n(a_1 + a_n)$; the number of terms in the sum is $2j_2 + 1$ when $j_1 \geq j_2$.

Example 4.3 Using the Clebsch–Gordan series

What angular momentum states can arise from a system with two sources of angular momentum, one with $j_1 = \frac{1}{2}$ and the other with $j_2 = \frac{3}{2}$? Specify the states.

Method. Use the Clebsch–Gordan series in eqn 4.43 to find the highest and lowest values of j first, and then complete the series. The composite system has $(2j_1 + 1)(2j_2 + 1)$ states, which may either be specified as $|j_1m_{j1}; j_2m_{j2}\rangle$ or as $|j_1j_2; jm_j\rangle$.

Answer. The highest and lowest values of j are $\frac{1}{2} + \frac{3}{2} = 2$ and $|\frac{1}{2} - \frac{3}{2}| = 1$, respectively. So the complete Clebsch–Gordan series is

$$j = 2, 1$$

A specification of the $4 \times 2 = 8$ states in the uncoupled representation is:

$$|\tfrac{1}{2}, +\tfrac{1}{2}; \tfrac{3}{2}, +\tfrac{3}{2}\rangle \quad |\tfrac{1}{2}, +\tfrac{1}{2}; \tfrac{3}{2}, +\tfrac{1}{2}\rangle \quad |\tfrac{1}{2}, +\tfrac{1}{2}; \tfrac{3}{2}, -\tfrac{1}{2}\rangle \quad |\tfrac{1}{2}, +\tfrac{1}{2}; \tfrac{3}{2}, -\tfrac{3}{2}\rangle$$
$$|\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{3}{2}, +\tfrac{3}{2}\rangle \quad |\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{3}{2}, +\tfrac{1}{2}\rangle \quad |\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{3}{2}, -\tfrac{1}{2}\rangle \quad |\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{3}{2}, -\tfrac{3}{2}\rangle$$

The alternative specification, in the coupled representation, is

$$|\tfrac{1}{2}, \tfrac{3}{2}; 2, +2\rangle \quad |\tfrac{1}{2}, \tfrac{3}{2}; 2, +1\rangle \quad |\tfrac{1}{2}, \tfrac{3}{2}; 2, 0\rangle \quad |\tfrac{1}{2}, \tfrac{3}{2}; 2, -1\rangle$$
$$|\tfrac{1}{2}, \tfrac{3}{2}; 2, -2\rangle \quad |\tfrac{1}{2}, \tfrac{3}{2}; 1, +1\rangle \quad |\tfrac{1}{2}, \tfrac{3}{2}; 1, 0\rangle \quad |\tfrac{1}{2}, \tfrac{3}{2}; 1, -1\rangle$$

Comment. The eight states in the coupled representation are linear combinations of the eight states in the uncoupled representation. We explore the relation between them in Section 4.13.

Exercise 4.3. Repeat the question for $j_1 = 1$ and $j_2 = 2$.

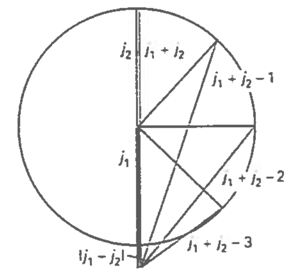


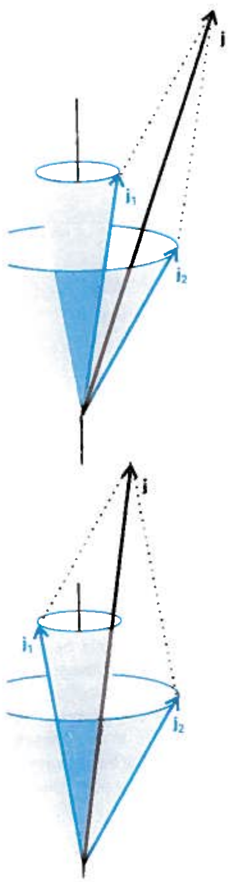
Fig. 4.5 The triangle condition corresponding to the Clebsch–Gordan series. The allowed values of j are those for which lines of length j, j_1, and j_2 can be used to form a triangle.

The Clebsch–Gordan series can be expressed in a simple pictorial way Suppose we are given rods of lengths j_1 and j_2 and are asked for the length j of the third side of a triangle that can be formed using these two rods (with al three lengths integers or half-integers). Then the answer would be precisel those given by the Clebsch–Gordan series (Fig. 4.5). For example, $j_1 =$ and $j_2 = 1$ require rods of lengths $j = 2, 1, 0$ to form a triangle. Althougl the **triangle condition** is no more than a simple and helpful mnemonic, i does suggest that angular momenta in quantum mechanics do in some respect behave like vectors and that the total angular momentum can be regarded as th resultant of the contributing momenta. The exploration of this point leads to th vector model of coupled angular momenta.

4.11 The vector model of coupled angular momenta

The vector model of coupled angular momentum is an attempt to represen pictorially the features of coupled angular momenta that we have deduce from the commutation relations. The approach gives insight into the signifi cance of various coupling schemes and is often a helpful guide to th imagination: it puts visual flesh on the operator bones.

The features that the vector diagrams of coupled momenta must express ar



4.6 Two possible states of total ular momentum that can arise from two :ified contributing momenta with ntum numbers j_1 and j_2. The relative ntations of the contributing momenta heir cones determine the total ;nitude.

as follows:

1. The length of the vector representing the total angular momentum is $\{j(j+1)\}^{1/2}$, with j one of the values permitted by the Clebsch–Gordan series.
2. This vector must lie at an indeterminate angle on a cone about the z-axis (because j_x and j_y cannot be specified if j_z has been specified).
3. The lengths of the contributing angular momentum vectors are $\{j_1(j_1+1)\}^{1/2}$ and $\{j_2(j_2+1)\}^{1/2}$. These lengths have definite values even when j is specified.
4. The projection of the total angular momentum on the z-axis is m_j; in the coupled picture (in which j is specified), the values of m_{j1} and m_{j2} are indefinite, but their sum is equal to m_j.
5. In the uncoupled picture (in which j is not specified), the individual components m_{j1} and m_{j2} may be specified, and their sum is equal to m_j.

The diagrams in Fig. 4.6 and Fig. 4.7 capture these points. In Fig. 4.6(a) is shown one of the states of the uncoupled picture: both m_{j1} and m_{j2} are specified, but there is no indication of the relative orientation of j_1 and j_2 apart from the fact that they lie on their respective cones. The total angular momentum is therefore indeterminate, for it could be either of the resultants shown in (a) or (b) or anything in between. In Fig. 4.7 is shown one of the states of the coupled picture. Now the resultant, the total angular momentum, has a well-defined magnitude and resultant on the z-axis, but the individual components m_{j1} and m_{j2} are indeterminate. It is important not to think of the vectors as precessing around their cones: at this stage the vector model is a display of possible but unspecifiable orientations.

An important example, and one that we shall encounter many times in later chapters, is the case of two particles with spin $s=\frac{1}{2}$, such as two electrons or two protons. For each particle, $s=\frac{1}{2}$ and $m_s=\pm\frac{1}{2}$. In the uncoupled picture, the electrons may be in any of the four states

$$\alpha_1\alpha_2 \qquad \alpha_1\beta_2 \qquad \beta_1\alpha_2 \qquad \beta_1\beta_2$$

These four states are illustrated in Fig. 4.8. The individual angular momenta lie at unspecified positions on their cones and the total angular momentum is indeterminate.

Now consider the coupled picture. The triangle condition (or the Clebsch–Gordan series) tells us that the total spin S (upper-case letters are used to denote the angular momenta of collections of particles) can take the values 1 and 0. When $S=0$, there is only one possible value of its z-component, namely 0, corresponding to $M_S=0$. Such a coupled state is called a **singlet**. When $S=1$, $M_S=+1, 0, -1$, and so this coupled arrangement is called a **triplet**.

The vector model of the triplet is shown in Fig. 4.9. The cones have been drawn to scale, and several points should be apparent. One is that to arrive at a resultant corresponding to $S=1$ (of length $\sqrt{2}$) using component vectors corresponding to $s=\frac{1}{2}$ (of length $\frac{1}{2}\sqrt{3}$), the vectors must lie at a definite angle relative to one another. In fact, they must lie in the same plane, as shown in the illustration, for only that orientation results in a vector of the correct length. Note that although spins are said to be 'parallel' in a triplet state (and represented ↑↑), they are in fact at an acute angle (of close to 70°). The two spins

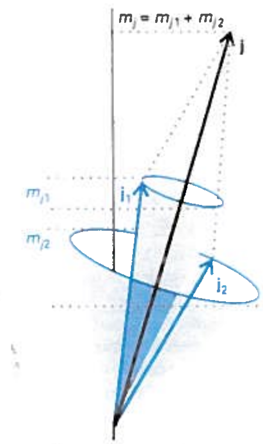


Fig. 4.7 If the two contributing momenta are locked together so that they give rise to a specified total, the projections of the contributing momenta span a range (as depicted by the vertical bars) and although their sum can be specified, their individual values cannot be specified.

make the same angle to one another in the states with $M_S=\pm1$; that is nece sary if they are to have the same resultant.

The vector model of the singlet must represent a state in which the sp angular momentum vectors sum to give a zero resultant (Fig. 4.10). It is cle from the illustration that the two spins are truly antiparallel (↑↓) in this state. in the triplet states, only the relative orientation of the vectors is fixed; the abs lute orientation is completely indeterminate.

4.12 The relation between schemes

The state $|j_1j_2;jm_j\rangle$ is built from all values of m_{j1} and m_{j2} such th $m_{j1}+m_{j2}=m_j$. This remark suggests that it should be possible to express t coupled state as a sum over all the uncoupled states $|j_1m_{j1};j_2m_{j2}\rangle$ that confo to $m_{j1}+m_{j2}=m_j$. It follows that we should be able to write

$$|j_1j_2;jm_j\rangle = \sum_{m_{j1},m_{j2}} C_{m_{j1}m_{j2}}|j_1m_{j1};j_2m_{j2}\rangle \qquad (4$$

The coefficients $C_{m_{j1}m_{j2}}$ are called **vector coupling coefficients**. Alternati names are 'Clebsch–Gordan coefficients', 'Wigner coefficients', and (in slightly modified form), the '3*j*-symbols'.

We shall illustrate the use of vector coupling coefficients by considering t singlet and triplet states of two spin-$\frac{1}{2}$ particles. The values are set out in Tal 4.1 (more values for other cases will be found in Appendix 2). The values in t

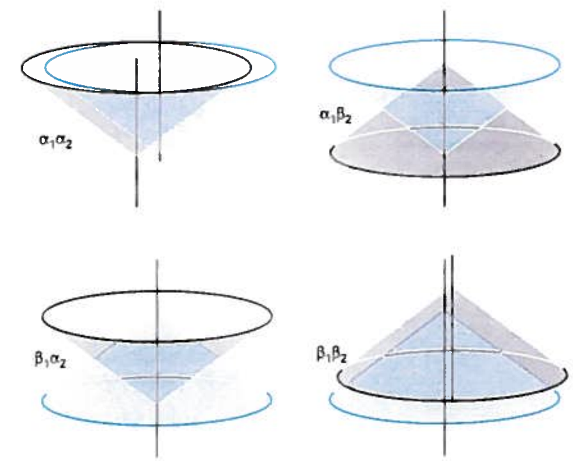


Fig. 4.8 The four uncoupled states of a system consisting of two spin-$\frac{1}{2}$ particles (such as electrons).