Relationship between Einstein Coefficients

Lets analyze the 2 level system in the limit of low density $n_e \to 0$.

$$n_1 B_{12} u_\nu = n_2 (A_{21} + B_{21} u_\nu)$$

Solve for u_ν the energy density

Divide by $n_2 B_{21}$:

$$\frac{n_1 B_{12}}{n_2 B_{21}} u_\nu - u_\nu = \frac{A_{21}}{B_{21}}$$

$$u_\nu = \frac{A_{21}/B_{21}}{\frac{B_{12}}{B_{21}} \cdot \frac{n_1}{n_2} - 1}$$

n_1/n_2 is related through Boltzmann's Equation

$$u_\nu = \frac{A_{21}/B_{21}}{\frac{B_{12}}{B_{21}} \frac{g_1}{g_2} e^{h\nu/kT} - 1}$$

But $u_\nu = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}$ for a blackbody

$$A_{21} = \frac{8\pi h \nu^3}{c^3} B_{21} \Rightarrow B_{21} = \frac{c^3}{8\pi h \nu^3} A_{21}$$

$$\frac{B_{12}}{B_{21}} \frac{g_1}{g_2} = 1 \Rightarrow g_1 B_{12} = g_2 B_{21}$$

CAN NOW CONVERT B's to A's.

EINSTEIN A DERIVATION

The exact equation depends on the "multipole" of radiation. E&M radiation is generated by acceleration of charge. The strongest, most common "multipole" is Electric Dipole

SEMI-Classical derivation:

CONSIDER an oscillating electric dipole:

$$d = e \cdot x(t) = e x_0 \cos \omega t$$

$\Rightarrow$ acceleration is $\ddot{x}(t) = x_0 \omega^2 \cos \omega t$

E&M theory - the power radiated into 4π steradians from an oscillating charge is:

$$P(t) = \frac{2}{3} \frac{e^2 \ddot{x}(t)}{c^3}$$

Let's average over one period of the oscillation:

$$\langle P \rangle = \frac{2}{3c^3} \cdot e^2 x_0^2 (2\pi\nu)^4 \cdot \frac{\int_0^{2\pi/\omega} \cos^2 \omega t \, dt}{\int_0^{2\pi/\omega} dt}$$

$$\int_0^{2\pi/\omega} \cos^2 \omega t \, dt = \int_0^{2\pi} \cos^2 t' \frac{dt'}{\omega} = \frac{1}{2\omega}\left(t' + \sin t' \cos t'\right)\Big|_0^{2\pi}$$

$$= \frac{1}{2\omega}(2\pi + 0 - 0 - 0) = \frac{\pi}{\omega}$$

$$\langle P \rangle = \frac{2}{3c^3} \cdot 16\pi^4 \nu^4 \cdot \frac{\pi/\omega}{2\pi/\omega} \cdot e^2 x_0^2$$

Define the average electric dipole as $\bar{\mu}_e = \frac{e x_0}{2}$

$$\langle P \rangle = \frac{64\pi^4 \nu^3}{3c^3} \cdot \left(\bar{\mu}_e\right)$$

Now considering that from quantum point of view

$$\langle P \rangle = h\nu A_{ue}$$

then

notice ν^3 dependence

electric dipole moment operator

$$A_{ue} = \frac{64\pi^4 \nu^{3}}{3hc^3} |\mu_{ue}|^2 \qquad \mu \sim \iiint \psi^* \hat{\mu}_e \psi \, dV$$

$[s^{-1}]$

electric dipole moment matrix element

Valid for electric dipole transitions.

$\circledast\circledast$ Note there are other expressions from magnetic dipole, electric quadrupole, etc. transitions...

ED $\sim \nu^3$
MD $\sim \nu^3$
EQ $\sim \nu^5$ etc.

A NOTE ON UNITS of $|\mu_{ue}|^2$

μ_e = electric dipole has units of $1 \text{ Debye} = 1\text{ D} = 10^{-18}$ esu

electrostatic units in cgs

$$1 \text{ esu} = 1 \; g^{1/2} cm^{3/2} s^{-1}$$ in cgs fundamental units

1 esu = 2 point charges spaced 1 cm apart produce 1 esu if the force between them is 1 dyne.

$\Rightarrow$ $1 \text{ esu} = 1 \text{ cm} \cdot \sqrt{\text{dyne}} = 1 \; g^{1/2} cm^{3/2} s^{-1}$