

Relationship between Einstein Coefficients

Let's analyze the 2 level system in the limit of low density $n_c \rightarrow 0$.

$$n_1 B_{12} u_\nu = n_2 (A_{21} + B_{21} u_\nu)$$

Solve for u_ν the energy density

Divide by $n_2 B_{21}$

$$\frac{n_1 B_{12}}{n_2 B_{21}} u_\nu - u_\nu = \frac{A_{21}}{B_{21}}$$

$$u_\nu = \frac{A_{21}/B_{21}}{\frac{B_{12}}{B_{21}} \frac{n_1}{n_2} - 1}$$

n_1/n_2 is related through Boltzmann's Equation

$$u_\nu = \frac{A_{21}/B_{21}}{\frac{B_{12}}{B_{21}} \frac{g_1}{g_2} e^{h\nu/kT} - 1}$$

But $u_\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}$ for a blackbody

$$A_{21} = \frac{8\pi h\nu^3}{c^3} B_{21} \Rightarrow B_{21} = \frac{c^3}{8\pi h\nu^3} A_{21}$$

$$\frac{B_{12}}{B_{21}} \frac{g_1}{g_2} = 1 \Rightarrow g_1 B_{12} = g_2 B_{21}$$

CAN NOW CONVERT B'S to A'S.

EINSTEIN A DERIVATION

The exact equation depends on the "multipole" of radiation. E&M radiation is generated by acceleration of charge. The strongest, most common "multipole" is Electric Dipole

Semi-classical derivation

Consider an oscillating electric dipole:

$$d = e \cdot x(t) = e x_0 \cos \omega t$$

$$\Rightarrow \text{acceleration } \ddot{x}(t) = x_0 \omega^2 \cos \omega t$$

E&M theory - the power radiated into 4π steradians from an oscillating charge is:

$$P(t) = \frac{2}{3} \frac{e^2 \ddot{x}^2(t)}{c^3}$$

Let's average over one period of the oscillation:

$$\langle P \rangle = \frac{2}{3c^3} \cdot e^2 x_0^2 (2\pi\nu)^4 \cdot \frac{\int_0^{2\pi/\omega} \cos^2 \omega t \, dt}{\int_0^{2\pi/\omega} dt}$$

$$\begin{aligned} \int_0^{2\pi/\omega} \cos^2 \omega t \, dt &= \int_0^{2\pi} \cos^2 t' \frac{dt'}{\omega} = \frac{1}{2\omega} (t' + \sin t' \cos t') \Big|_0^{2\pi} \\ &= \frac{1}{2\omega} (2\pi + 0 - 0 - 0) = \frac{\pi}{\omega} \end{aligned}$$

$$\langle P \rangle = \frac{2}{3c^3} \cdot 16\pi^4 \nu^4 \cdot \frac{\pi/\omega}{2\pi/\omega} \cdot e^2 x_0^2$$

Define the average electric dipole as $\bar{\mu}_e = \frac{e x_0}{2}$

$$\langle P \rangle = \frac{64\pi^4 \nu^3}{3c^3} \cdot (\bar{\mu}_e)$$

Now considering that from quantum point of view

$$\langle P \rangle = h\nu A_{ue}$$

then

$$A_{ue} = \frac{64\pi^4 \nu^3}{3hc^3} |\mu_{ue}|^2$$

[s⁻¹]

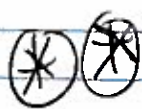
Valid for electric dipole transitions.

Notice ν^3 dependence

electric dipole moment operator

$$\mu = \int \psi^* \hat{\mu}_e \psi dV$$

electric dipole moment matrix element



Note there are other expressions from magnetic dipole, electric quadrupole, etc. transitions...

$$\begin{aligned} \text{ED} &\sim \nu^3 \\ \text{MD} &\sim \nu^3 \\ \text{EQ} &\sim \nu^5 \text{ etc.} \end{aligned}$$

A NOTE ON UNITS OF $|\mu_{ue}|^2$

$\mu_e = \text{electric dipole}$ has units of 1 Debye = 1 D = 10⁻¹⁸ esu
electrostatic units
in cgs

$$1 \text{ esu} = 1 \text{ g}^{1/2} \text{ cm}^{3/2} \text{ s}^{-1} \text{ in cgs fundamental units}$$

1 esu = 2 point charges spaced 1 cm apart produce 1 esu if the force between them is 1 dyne.

$$\Rightarrow 1 \text{ esu} = 1 \text{ cm} \cdot \sqrt{\text{dyne}} = 1 \text{ g}^{1/2} \text{ cm}^{3/2} \text{ s}^{-1}$$