Electrostatic Bremsstrahlung - "Free-Free" Radiation

In plasma, e^- and ion are accelerated as they pass each other on scales < Debye length of plasma

The power radiated from an accelerated ($\dot{v}$) charge is given classically from Larmor's formula:

$$P = \frac{2}{3}\frac{e^2 \dot{v}^2}{c^3} \quad \text{erg} \cdot \text{s}^{-1}$$

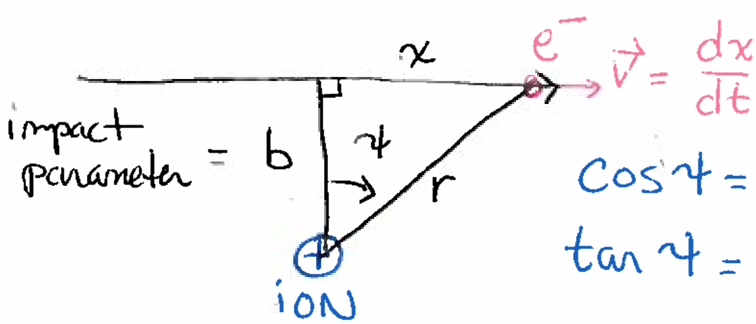
{ cgs NOTE: $e \sim 4.8 \times 10^{-10}$ statcoloumb

Since $F = ma$ then $a = \dot{v} \sim \frac{1}{m}$. $P \sim \dot{v}^2 \sim \frac{1}{m^2}$

This implies radiation from e^- much more important than radiation from ions because $P_e / P_{ions} \sim (m_p/m_e)^2 \sim 4 \times 10^6$.

Furthermore, interactions between identical particles are negligible because $\dot{v}_1 = -\dot{v}_2$ and the radiated electric fields are equal in magnitude but opposite in sign, so the net radiated field $\to 0$ at distances $\gg$ collision impact parameter.

$\Rightarrow$ ONLY e^- - ion collisions important
AND ONLY e^- radiate signifantly



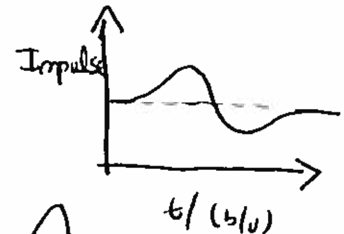
$$\cos\psi = \frac{b}{r} \Rightarrow r = \frac{b}{\cos\psi}$$

$$\tan\psi = \frac{x}{b}$$

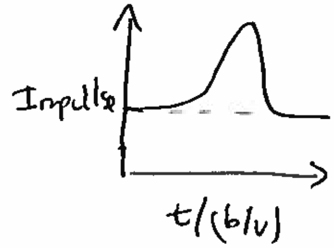
Define $\tau \approx \frac{b}{v}$ = "Collision time"

Since e^- moving so fast, we can approximate its trajectory as a straight line.

$$F_{\parallel} = m_e \dot{v}_{\parallel} = \frac{-Ze^2}{r^2}\sin\psi = \frac{-Ze^2 \sin\psi \cos^2\psi}{b^2}$$



$$F_{\perp} = m_e \dot{v}_{\perp} = \frac{Ze^2}{r^2}\cos\psi = \frac{Ze^2 \cos^3\psi}{b^2}$$

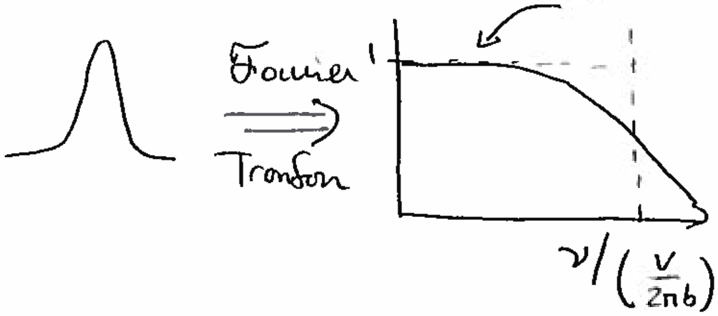


single peak and narrow width ~ b/v

$\dot{V}_{\parallel}$ pulse is sinusoidal with $\omega = 2\pi\nu \sim \frac{1}{\tau} = \frac{V}{b} \Rightarrow$ high ν (IR) radiation

$\dot{V}_{\perp}$ pulse is single peaked (narrow width ~ $\frac{b}{v}$)

Fourier transform is $\simeq$ flat up to a $\nu \sim \frac{V}{2\pi b}$



$\Rightarrow$ Frequency response is nearly const with ν at low frequency.

For rest of problem, we will only consider $\dot{V}_{\perp}$

Our derivation will get correct functional dependence, but will not get constants correct. To do this problem correctly need to:

1. Calculate electric field $E(t)$ of far field electric dipole
2. Take Fourier Transform $|\tilde{E}_\nu|$
3. Integrate $|\tilde{E}_\nu|^2$ over solid angle to get W_ν $\quad$ erg · Hz^{-1}

Solution is :

$$W_\nu = \frac{4\pi^2}{3}\frac{Z^2 e^6}{m_e^2 c^3 b^2 v^2} e^{-4\pi\nu b/V}$$

Let's work through the simplier derivation ...

Power radiated from accelerated $\dot{v}_\perp$ charge:

$$P_\perp = \frac{2}{3}\frac{e^2 \dot{v}_\perp^2}{c^3} = \frac{2e^2}{3c^3}\frac{z^2 e^4}{m_e^2}\left(\frac{\cos^3\psi}{b^2}\right) \quad \text{erg} \cdot s^{-1}$$

The total energy emitted by the pulse is:

$$W = \int_{-\infty}^{\infty} P_\perp \, dt \quad \text{erg}$$

Strategy: use geometry and approximately const e^- v to turn $dt \to d\psi$

$$v = \frac{dx}{dt} = b\,\frac{d(\tan\psi)}{dt} = b\,\frac{\sec^2\psi\, d\psi}{dt} \Rightarrow dt = \frac{b}{v}\frac{d\psi}{\cos^2\psi}$$

$$W \sim \frac{z^2 e^6}{c^3 m_e^2 b^3 v} \int_0^{\pi/2} \cos^4\psi \, d\psi$$

$3\pi/16 \Rightarrow$ just a constant

Now, since we know that Fourier transform of pulse profile was nearly flat up to $\nu_{max} = \frac{1}{2\pi\tau} = \frac{v}{2\pi b}$

Let's approximate W_ν (average energy emitted per unit frequency)

$$W_\nu \simeq \frac{W}{\nu_{max}} \simeq \frac{W}{\left(\frac{v}{2\pi b}\right)} \sim \frac{z^2 e^6}{c^3 m_e^2 \cancel{b^3}^{\,2} v} \cdot \frac{\cancel{b}}{v}$$

$$W_\nu \sim \frac{z^2 e^6}{c^3 m_e^2 b^2 v^2} \quad \text{erg} \cdot \text{Hz}^{-1}$$

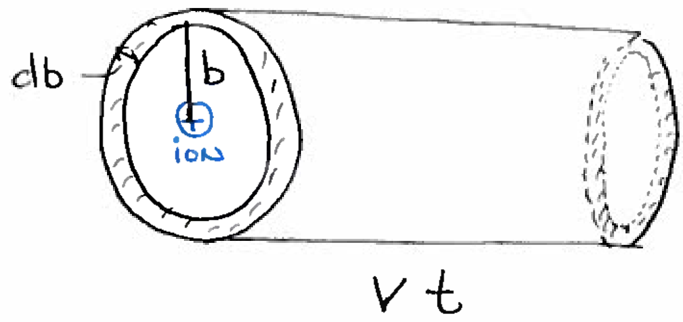
NOTE: $W_\nu(b, v)$

Valid for $\nu < \nu_{max} \approx 10^{14}$ Hz in HII regions.

This is for a single interaction $\Rightarrow$

Need to $\int_b \int_v$ to get j_ν ...

In LTE $(KE)_{ions} = (KE)_{e^-}$ $\Rightarrow$ ions are essentially stationary during e^- interaction



The number of e^- passing a single ion per unit time at impact parameters $b + db$ with speed $v + dv$ is

THINK: $\frac{1}{t} = n \sigma v$

$$n_e \, (2\pi b \, dv) \; v \, f(v) \, dv \qquad s^{-1}$$

speed distribution of e^-
IN LTE = Maxwell-Boltzmann Distribution $\int f(v) dv = 1$

This is only for 1 target ion. To get total number of "collisions" per unit volume multiply by n_i cm^{-3}

The free-free emissivity (spectral power per 4π Ster):

$$j_\nu = \frac{1}{4\pi} \int_{b=0}^{\infty} \int_{v=0}^{\infty} W_\nu(v,b) \, n_i n_e \, 2\pi b \, db \; v \, f(v) \, dv$$

$$\sim \frac{Z^2 e^6 n_e n_i}{c^3 m_e^2} \int_{v=0}^{\infty} \frac{f(v)}{v} dv \int_{b=0}^{\infty} \frac{db}{b}$$

Let's evaluate these 2 integrals:

First, the velocity integral:

$$f(v) = 4\pi \left(\frac{m_e}{2\pi kT}\right)^{3/2} e^{-m_e v^2/2kT} \cdot v^2$$

We have to consider the additional issue of discreetness of photon energies: $h\nu = \frac{1}{2} m_e v_{min}^2$ sets $\underline{v_{min}}$

In other words, e^- must have at least v_{min} to produce photons of energy ν $\rightarrow$ this set the lower limit to $\int$.

$$\int_{v_{min}}^{\infty} \frac{f(v)}{v}\, dv \sim \left(\frac{2m_e}{\pi kT}\right)^{1/2} e^{-m_e v_{min}^2/2kT}$$

$$\sim \left(\frac{2m_e}{\pi kT}\right)^{1/2} e^{-h\nu/kT}$$

At radio ν, $e^{-h\nu/kT} \approx 1$ for $T = 10^4$ K HII regions.
This term only important at high ν.

Notice the integral over db diverges logarithmically!
This means we have to chose physically meaningful limits b_{min} & b_{max} $\quad \int_{b_{min}}^{b_{max}} \frac{db}{b} = \ln\left(\frac{b_{max}}{b_{min}}\right)$

$\underline{b_{min}}$	$\underline{b_{max}}$
Classical: Maximum momentum $m_e \Delta v$ is 2x initial momentum	Classical: Largest b that can generate significant power at ν:
$m_e \Delta v = \int_{-\infty}^{\infty} F_\perp\, dt = \frac{2Ze^2}{bv}$	$b_{max} \sim \frac{v}{\omega} = \frac{v}{2\pi\nu}$ $\quad$ see Fourier Transform Figure
$\Rightarrow b_{min} \sim Ze^2/m_e v^2$	Debye length $\lambda_D \simeq \left(\frac{kT}{4\pi n_e e^2}\right)^{1/2}$
Quantum: Heisenburg Uncertainty	Characteristic length scale for charge shielding in a plasma.
$b_{min} \simeq \hbar/m_e v$	

For HII regions, classical limits are appropriate:

$$\frac{b_{max}}{b_{min}} \sim \frac{v}{2\pi\nu} \cdot \frac{m_e v^2}{Z e^2} \sim \frac{1}{\nu}$$

$$\Rightarrow \int_{b_{min}}^{b_{max}} \frac{db}{b} \sim \ln\left(\frac{1}{\nu}\right) \sim \nu^{-0.1}$$ is reasonable approximation.

The more precise calculation takes quantum effects into account and averages over $f(v)$. It is called the gaunt factor

$$\overline{g_{ff}} \simeq \frac{\sqrt{3}}{\pi}\left[\ln \frac{(2kT)^{3/2}}{\pi Z e^2 m_e^{1/2} \nu} - \frac{5\gamma}{2}\right]$$

Euler's Constant

see Draine Chapter 10

$$\simeq 6.155\,(Z\,\nu_{GHz})^{-0.118}\,(T/10^4 K)^{0.177}$$

So the emissivity coefficient is then: Using results from "correct" derivation now

$$j_\nu = \frac{8}{3}\left(\frac{2\pi}{3}\right)^{1/2} \frac{Z^2 e^6}{m_e^2 c^3}\left(\frac{m_e}{kT}\right)^{1/2} n_e n_i\, e^{-h\nu/kT}\, \overline{g_{ff}}$$

$erg \cdot s^{-1} \cdot cm^{-3} \cdot ster^{-1} \cdot Hz^{-1}$

Notice at low ν (when $e^{-h\nu/kT} \sim 1$) that

$$j_\nu \sim \nu^{-0.118}$$ ← very weak frequency dependence.

This is because as e^- flies by ion, its acceleration is like a sharp pulse. The Fourier transform of a sharp pulse is $\approx$ flat with frequency.