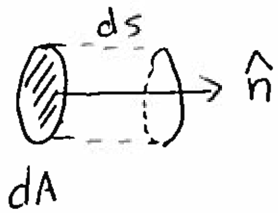
Radiative Transfer

ASTR 300B



Consider I_ν passing through dA in direction $\hat{n}$. If over a distance ds there is a change of dI_ν, then the Energy changes by:

$$dE = dI_\nu \, dA \, d\Omega \, dt \, d\nu$$

(NOTE: $\hat{n}$ is $\perp$ dA so NO $\cos\theta$)

This change in energy must equal the change in energy due to emission − absorption in the length ds:

$$dE = dE_{emit} - dE_{abs}$$

(We are ignoring scattering)

We define 2 new quantities to describe absorption / emission

α_ν = Absorption Coefficient cm^{-1} = Mean number of photons absorbed per cm along ds

j_ν = Emissivity Coefficient $erg \cdot s^{-1} \cdot cm^{-3} \cdot ster^{-1} \cdot Hz^{-1}$ = Amount of energy emitted per Hz at frequency ν into solid angle $d\Omega$ from a unit volume per unit time

$$dE_{abs} = \alpha_\nu I_\nu \, ds \, dA \, d\Omega \, dt \, d\nu$$

$$dE_{emit} = j_\nu \, ds \, dA \, d\Omega \, dt \, d\nu$$

Plugging these equations into $dE = dE_{emit} - dE_{abs}$:

$$dI_\nu \, \cancel{dA} \, \cancel{d\Omega} \, \cancel{dt} \, \cancel{d\nu} = j_\nu \, ds \, \cancel{dA} \, \cancel{d\Omega} \, \cancel{dt} \, \cancel{d\nu} - \alpha_\nu I_\nu \, ds \, \cancel{dA} \, \cancel{d\Omega} \, \cancel{dt} \, \cancel{d\nu}$$

1D Radiative Transfer Equation:

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$$

(ignoring scattering)

$erg \cdot s^{-1} \cdot cm^{-3} \cdot ster^{-1} \cdot Hz^{-1}$

If we divide both sides by α_ν, we define 2 new quantities:

$$\frac{dI_\nu}{\alpha_\nu\, ds} = \frac{j_\nu}{\alpha_\nu} - I_\nu \implies \frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

optical depth

Source Function

① Optical Depth : τ

$$d\tau_\nu \equiv \alpha_\nu\, ds$$

$$\tau_\nu = \int_0^L \alpha_\nu(s)\, ds$$

total path length

$cm^{-1} \cdot cm$ = UNITLESS

Optical Depth is unitless

② Source Function : S_ν

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu}$$

Same units as I_ν!

$erg \cdot s^{-1} \cdot cm^{-2} \cdot ster^{-1} \cdot Hz^{-1}$

Consider the case of thermodynamic equilibrium:

Now $I_\nu = B_\nu(T)$ Planck Function

Thermodynamic equilibrium $\Rightarrow \frac{dI_\nu}{ds} = 0 \Rightarrow I_\nu = B_\nu(T) = \text{CONST}$

$$0 = j_\nu - \alpha_\nu B_\nu(T)$$

$$B_\nu(T) = \frac{j_\nu}{\alpha_\nu}$$ Kirchoff's Law

Means emission & absorption are in balance $\left(\frac{dI_\nu}{ds} = 0\right)$ with a source function = Planck Function.

Formal 1D Solution with Emission & Absorption

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

This is a linear 1st order diffy Q

Integrating factor is $e^{\int_0^{\tau_\nu} +1\, d\tau_\nu'} = e^{+\tau_\nu}$

$$e^{\tau_\nu}\frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = e^{\tau_\nu} S_\nu$$

NOTE: $\frac{d}{d\tau_\nu}\left[e^{\tau_\nu} I_\nu\right] = e^{\tau_\nu}\frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu$

Let's Define temporarily $\mathcal{I} \equiv e^{\tau_\nu} I_\nu$:

$$\int_{\mathcal{I}(0)}^{\mathcal{I}(\tau_\nu)} d\mathcal{I} = \int_0^{\tau_\nu} e^{\tau_\nu'} S_\nu(\tau_\nu')\, d\tau_\nu'$$

$$\mathcal{I}(\tau_\nu) = \mathcal{I}(0) + \int_0^{\tau_\nu} e^{\tau_\nu'} S_\nu(\tau_\nu')\, d\tau_\nu'$$

TOTAL OPTICAL DEPTH = CONST

① Substitute for $\mathcal{I} = e^{\tau_\nu} I_\nu$ and ② Multiply equation by $e^{-\tau_\nu}$

$$\boxed{I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')} S_\nu(\tau_\nu')\, d\tau_\nu'}$$

Consider case of $S_\nu(\tau_\nu') = \text{CONST} = S_\nu$

$$I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu} + S_\nu \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')}\, d\tau_\nu'$$

Let $x = -(\tau_\nu - \tau_\nu') \Rightarrow dx = d\tau_\nu'$

[Remember: τ_ν is a CONST = TOTAL OPTICAL DEPTH!]

$$\int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')}\, d\tau_\nu' = \int_{-\tau_\nu}^{0} e^{x}\, dx = 1 - e^{-\tau_\nu}$$

$$\boxed{I_\nu = I_\nu(0)\, e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right)}$$