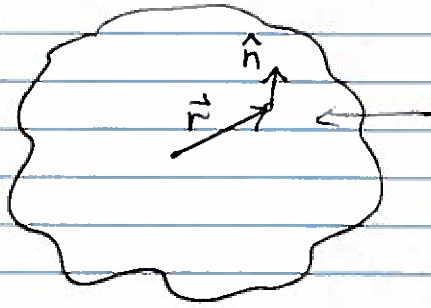
The atoms or molecules in the cloud also generate and absorb their own photons with $\mathcal{U}_\nu(T_{ex})$



At some point $(\vec{r})$ in the cloud the optical depth in a direction $\hat{n}$ is $\tau_\nu(\hat{n},\vec{r})$

The probability that a photon will escape (NOT BE ABSORBED!) is given by $P_{escape} = e^{-\tau}$

Averaging over all solid angles gives the escape probability

NOTE $\beta = 1$ optically thin all photons escape

$$\bar{\beta}_\nu(\vec{r}) = \int_\Omega e^{-\tau_\nu(\hat{n},\vec{r})} \frac{d\Omega}{4\pi}$$

$\beta = 0$ extremely optically thick (no photons escape)

The photons are emitted and absorbed through a line profile fctn.

$$\langle \beta(\vec{r}) \rangle = \int \phi_\nu \, \bar{\beta}(\vec{r}) \, d\nu$$

Escape probability formalism with the "on the spot" approximation means we assume photons absorbed at same point as emission (this is ok with assumption of a homogeneous cloud)

NOTE Real radiative transfer problem NON-LOCAL! = Challenge!

Calculating $\langle \beta \rangle$ for a homogeneous spherical cloud

$$\langle \beta \rangle_{cloud} = \frac{3}{4\pi R^3} \int_0^R \langle \beta(\vec{r}) \rangle 4\pi r^2 dr$$

where

$$\tau_0 = \frac{c^3}{8\pi\nu^3} \frac{A_{ul}}{4(2\pi)^{1/2}\sigma_v} n_u R \left(\frac{n_l}{n_u} \frac{g_u}{g_l} - 1 \right)$$

optical depth at line center from center of cloud to the surface

See slides for solutions to this problem ...

$u_\nu = u_\nu^0 e^{-\tau_{ex}} + u_\nu^{ex}(1 - e^{-\tau_\nu})$

Let's apply this to the 2 level problem:

$$I_\nu = I_\nu^0 e^{-\tau_\nu} + B_\nu(T_{ex})(1 - e^{-\tau_\nu})$$

Note also true for u_ν!

Divide by $\frac{c^2}{2h\nu^3}$

$$n_\gamma = n_\gamma^0 e^{-\tau_\nu} + n_\gamma^{ex}(1 - e^{-\tau_\nu})$$

$n_\gamma^0 = n_\gamma(T_{bg})$
$n_\gamma^{ex} = n_\gamma(T_{ex})$

$$\iint \frac{n_\gamma}{4\pi} d\Omega\, \phi_\nu\, d\nu = \boxed{\langle n_\gamma \rangle = \bar{n}_\gamma^0 \langle \beta \rangle + \bar{n}_\gamma^{ex}(1 - \langle \beta \rangle)}$$

this is how we modify n_γ terms.

Start with 2 lvl system with $\langle\beta\rangle = 1$

I'm going to drop $\langle\rangle$ and $\bar{n}$ ← bars to ease notation

$$n_\gamma = n_\gamma^0$$

$$\frac{dn_u}{dt} = n_c(n_1\gamma_{12} - n_2\gamma_{21}) - n_2 A_{21} - n_2 A_{21} n_\gamma^0 + n_1 \frac{g_2}{g_1} A_{21} n_\gamma^0$$

$$= n_c(\cdot) - n_2 A_{21} + n_1 A_{21} \frac{g_2}{g_1} n_\gamma^0 \left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right)$$

Now let $\beta < 1$:

$$\frac{dn_u}{dt} = n_c(\cdot) - n_2 A_{21} - n_2 A_{21}\left[n_\gamma^0 \beta + n_\gamma^{ex}(1-\beta)\right] + n_1 \frac{g_2}{g_1} A_{21}\left[n_\gamma^0 \beta + n_\gamma^{ex}(1-\beta)\right]$$

Algebra – let's collect together terms with β and $(1-\beta)$:

$$\frac{dn_u}{dt} = n_c(\cdot) - n_2 A_{21} - n_2 A_{21} n_\gamma^0 \beta + n_1 A_{21} \frac{g_2}{g_1} n_\gamma^0 \beta - n_2 A_{21} n_\gamma^{ex}(1-\beta) + n_1 A_{21} \frac{g_2}{g_1} n_\gamma^{ex}(1-\beta)$$

$$\frac{dn_u}{dt} = n_c(\cdot) - n_2 A_{21} + n_1 \beta A_{21} \frac{g_2}{g_1} n_\gamma^0 \left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right)$$

$$+ n_2 (1-\beta) A_{21} n_\gamma^{ex} \left(\frac{n_1}{n_2}\frac{g_2}{g_1} - 1\right)$$

Look carefully at this term

$$\frac{n_1}{n_2}\frac{g_2}{g_1} - 1 = ? = e^{h\nu/kT_{ex}} - 1 = \frac{1}{n_\gamma^{ex}} \; !$$

Boltzmann $\quad \frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-h\nu/kT_{ex}}$

$$\Rightarrow e^{h\nu/kT_{ex}} = \frac{n_1}{n_2}\frac{g_2}{g_1}$$

Substituting back into our equation:

$$\frac{dn_u}{dt} = n_c(\cdot) \underbrace{- n_2 A_{21} + n_2(1-\beta)A_{21}\cancel{n_\gamma^{ex}}\left(\frac{1}{\cancel{n_\gamma^{ex}}}\right)}_{\text{simplify}} + n_1 \beta A_{21} \frac{g_2}{g_1} n_\gamma^{o}\left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right)$$

$$\cancel{-n_2 A_{21}} + \cancel{n_2 A_{21}} - n_2 \beta A_{21}$$

$$\frac{dn_u}{dt} = n_c(\cdot) - n_2 \beta A_{21} + n_1 \beta A_{21} \frac{g_2}{g_1} n_\gamma^{o}\left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right)$$

This is identical to 2 level definition above with A_{21} replaced with βA_{21}

The effect of stim emission & absorption at $u_\nu(T_{ex})$ is to "effectively decrease" the spontaneous emission rate by β

NOTE - if $\beta = 1$ (optically thin) we recover the original equation!

Critical Density: $\quad n_{crit} = \frac{\beta A_{ue}}{\gamma_{ue}} \leftarrow$ accounts for optical depth effects ("line trapping")

For large τ, $\beta \sim 1/\tau$

$$n_{crit} = \frac{(A_{ue}/\tau)}{\gamma_{ue}}$$

For multi-lvl system, should Σ over all collision rates