Virial Theorem

ASTR 250
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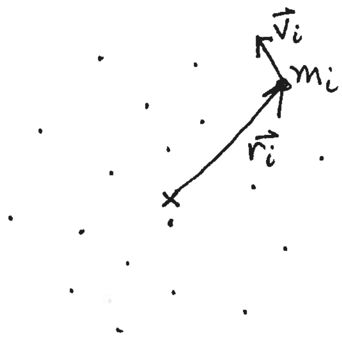
Derivation

ONE of the most important theorem's in astrophysics



imagine a system of N bound particles with masses, m_i, at at a distance $\vec{r}_i$ from the center of mass.

Each particle has momentum $\vec{p}_i = m_i \vec{v}_i$

Define "the Virial"

$$A \equiv \sum_{i=1}^{N} m_i \vec{v}_i \cdot \vec{r} = \sum_{i=1}^{N} \vec{p}_i \cdot \vec{r}_i$$

Now take $\frac{d}{dt}$ of A:

$$\frac{dA}{dt} = \sum_i \frac{d\vec{p}_i}{dt} \cdot \vec{r}_i + \vec{p}_i \cdot \underbrace{\frac{d\vec{r}_i}{dt}}_{\text{this is just } \vec{v}_i}$$

$$= \sum_i m_i \frac{d\vec{v}_i}{dt} \cdot \vec{r}_i + m\vec{v}_i \cdot \vec{v}_i$$

Note: $\vec{v}_i \cdot \vec{v}_i = v_i^2$

$$= \sum_i m_i \vec{a}_i \cdot \vec{r}_i + m v_i^2$$

Note: $\vec{F}_i = m\vec{a}_i$

$$= \sum_i \vec{F}_i \cdot \vec{r}_i + 2T$$

where T = kinetic energy of all particles $= \frac{1}{2} m v_i^2$

Now take the time average of $\frac{dA}{dt}$:

$$\left\langle \frac{dA}{dt} \right\rangle = \frac{1}{t}\int_0^t \frac{dA}{dt}\, dt$$

$$= \left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle + \langle 2T \rangle$$

If the system of partices is bound, then $\vec{r}_i$ and $\vec{p}_i$ remain finite. $\Rightarrow$ $\sum \vec{p}_i \cdot \vec{r}_i$ is finite $\Rightarrow$ so is it's time derivative.

The system of partizles is "virialized" if after a long enough time, t, $\left\langle \frac{dA}{dt} \right\rangle \to 0$.

Virialized $\Rightarrow$ $\langle 2T \rangle = -\left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle$

$\uparrow$ we can relate this to the potential energy of the system.

In general $\vec{F}_i = -\vec{\nabla}\,\mathcal{U}(r_i) = -\frac{d\mathcal{U}}{dr_i}$ for a purely radial force.

e.g. Gravity $F = -\frac{GMm}{r^2}$ and $\mathcal{U} = +\frac{GMm}{r}$ = Potential energy

If $\mathcal{U} \propto r^n$ then $F \propto r^{n-1}$ For gravity where $F \propto r^{-2}$ $\Rightarrow n = -1$.

$$\frac{d\mathcal{U}}{dr} \propto n r^{n-1}$$

so $$\frac{d\mathcal{U}}{dr} \cdot r \propto n \cdot r^n \propto n \cdot \mathcal{U}$$

so the term

$$-\sum_i \vec{F}_i \cdot \vec{r}_i = +\sum_i \frac{d\mathcal{U}}{dr_i} r_i$$

$$= +\sum_i n\,\mathcal{U}(r_i)$$

$$= +n\,\mathcal{U}$$

Back to the virial theorem:

$$\langle 2T \rangle = +n \langle \mathcal{U} \rangle$$

For Gravity $n=-1$ so

$$\boxed{\langle T \rangle = -\frac{1}{2} \langle \mathcal{U} \rangle}$$

The Virial Theorem applies when

1. Bound system of partices.
2. Force holding system together has purely radial dependence (e.g. $F_{grav} \propto r^{-2}$)
3. System is in "virial equilibrium" (or is "virialized") such that $\langle \frac{dA}{dt} \rangle \to 0$.

   Another way of saying the above statement is that over a long period of time, the Virial, A, is constant.