Stellar Energy Sources

ASTR 250

SPRING 2010

Last time we learned $\langle T \rangle = -\frac{1}{2}\langle U \rangle$ Virial thereom

and that $E_\odot^{life} = L_\odot \cdot t_{life} = 3.9 \times 10^{33}\,erg \cdot s^{-1} \cdot 5 \times 10^{9}\,yrs = 6 \times 10^{50}\,ergs$

is the total solar energy output over its lifetime

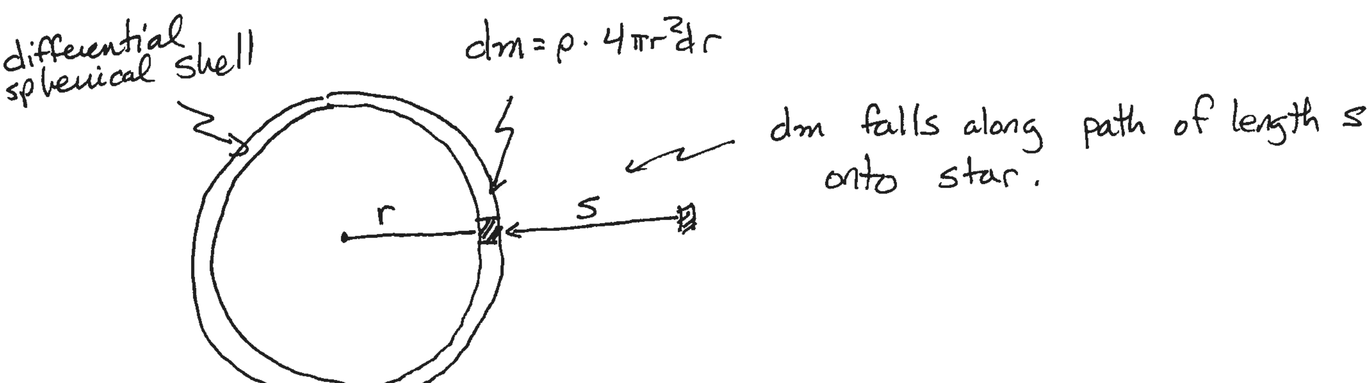
What supplies this energy?

(1.) Gravitational contraction of the Sun to its present radius, $R_\odot$?

Virial theorem $\Rightarrow$ $\frac{1}{2}$ of gravitational potential Energy of Sun, $U_\odot$ has been radiated away.

Lets calc. $U_\odot$:

Consider a star with mass, M and radius r with mass element $dm = \rho \cdot 4\pi r^2 dr$ falling onto the star from ∞.



The graviational energy release is

$$dE_{grav} = \int_{s=\infty}^{r} (\text{force on } dm) \cdot ds$$

$$\text{force} = dm \cdot \text{acceleration} = dm \cdot \frac{GM}{s^2}$$

so

$$dE_{grav} = \int_{s=\infty}^{r} \frac{GM}{s^2} \cdot 4\pi r^2 \rho \, dr \, ds$$

$$= -4\pi r^2 \rho \, dr \, \frac{GM}{s} \Big|_{\infty}^{r}$$

$$= -4\pi r^2 \rho \, dr \left(\frac{GM}{r} - \cancelto{0}{\frac{GM}{\infty}} \right)$$

$$dE_{grav} = - GM\rho 4\pi r \, dr$$

So, the total gravitational energy released by building up the star to radius R is given by:

$$E_{grav} = \int dE_{grav} = - \int_0^R GM\rho 4\pi r \, dr$$

Let assume the star is constant density : $\rho(r) = \rho = \text{CONST}$

Then $M(r) = \frac{4}{3}\pi r^3 \rho$

Substituting for M :

$$E_{grav} = - \int_0^R G \cdot \frac{4}{3}\pi r^3 \cdot \rho \cdot 4\pi r \rho \, dr$$

$$E_{grav} = -\frac{16 G \pi^2 \rho^2}{3} \int_0^R r^4 \, dr$$

$$= \frac{-16}{3} G\pi^2\rho^2 \cdot \left.\frac{r^5}{5}\right|_0^R = -\frac{16}{15}\pi^2 G \rho^2 R^5$$

Now, $\rho = \frac{3M}{4\pi R^3} \Rightarrow \rho^2 = \frac{9}{16}\frac{M^2}{\pi^2 R^6}$

$$E_{grav} = \frac{-\cancel{16}}{15}\cancel{\pi^2} G \cdot \frac{9}{\cancel{16}}\frac{M^2}{\cancel{\pi^2} R^6} \cdot R^5$$

$$\boxed{E_{grav} = -\frac{3}{5}\frac{GM^2}{R}}$$

Gravitational potential energy of a constant density sphere.

$$E^{grav}_{\odot} = -\frac{3}{5}\frac{6.67\times10^{-8}\,(2\times10^{33}\,g)^2}{7\times10^{10}\,cm} = 2.4\times10^{48}\ erg$$

SINCE $1/2$ of this energy was radiated away, it took the Sun t_{KH} to radiate that energy:

$$t_{KH} = \frac{1.2\times10^{48}\ erg}{3.9\times10^{37}\ erg\cdot s^{-1}} = 3.1\times10^{14}\,s \cdot \left(\frac{1\ yr}{3.1\times10^{7}\,s}\right) = 1\times10^{7}\ years.$$

This timescale is called the Kelvin-Helmholtz timescale

$$t_{KH} \ll t^{\odot}_{life} \ !!$$

(2.) What about chemical reactions?

Let's assume the sun consists of H and O in the optimum proportions to burned to form H_2O vapor:

H_2O has molecular weight $m(^{16}O) + 2m(^{1}H) = 18 \cdot m_H$

$$= 18 \cdot 1.67 \times 10^{-24}\,g$$

$$= 3.5 \times 10^{-23}\,g$$

If the entire sun burned

$$N_{H_2O} = \frac{2 \times 10^{33}\,g}{3.5 \times 10^{-23}\,g/H_2O\text{ molecule}} = 6 \times 10^{55}\ H_2O\text{ molecules}$$

Typical burning reactions create a visible flame, so let's conservatively estimate that $E = 10\,eV$ is released per reaction:

$$10\,eV \simeq 10^{-11}\,\text{erg/reaction}$$

$$E_{\text{chemical}} = 10^{-11}\,\text{erg/reaction} \cdot 6 \times 10^{55}\text{ reactions} \sim 6 \times 10^{44}\,\text{erg}$$

$$t_{\text{chemical}} = \frac{6 \times 10^{44}\,\text{erg}}{3.9 \times 10^{33}\,\text{erg} \cdot s^{-1}} \simeq 5000\text{ years}$$

$$t_{\text{chemical}} \lll t_{\odot}^{\text{life}}\ !$$

3. The answer? Nuclear Energy

$$E = mc^2$$

Nuclear fusion turns $4\,{}^{1}H \rightarrow 1\,{}^{4}He$

Let's look at their atomic masses:

$$m({}^{1}H) = 1.00794 \text{ amu}$$

$$m({}^{4}He) = 4.002602 \text{ amu}$$

$$4 \cdot m({}^{1}H) \neq m({}^{4}He) \;!!$$

This is because $\Delta E = \Delta m c^2$ – that Δm has been converted into energy.

$$\Delta m = m({}^{4}He) - 4 \cdot m({}^{1}H) = -0.0291 \text{ amu}$$
$$\Delta E = -\Delta m c^2 \cong 27 \text{ MeV for each reaction.}$$

The mass fraction is

$$\frac{\Delta m}{4} = 0.007 \Rightarrow$$ 0.7 % of mass of each proton is converted to Energy by fusion!

For <u>ENTIRE</u> Sun assuming it is entirely made of ${}^{1}H$:

$$E_{fusion} = \frac{\Delta m}{4} \cdot M_{\odot} c^2 = 0.007 \cdot 2\times10^{33}\,g \cdot (3\times10^{10}\,cm/s)^2$$
$$\simeq 1.3 \times 10^{52} \text{ erg!}$$

$$E_{fusion} > E_{\odot}^{life}$$ Nuclear fusion can power the Sun!

So what is a star ?

Gravitationally contained nuclear reactor that is in "Hydrostatic Equilibrium"

Hydrostatic Equilibrium = Balance between gas Pressure and gravity.

Equations of stellar structure :

1. hydrostatic equilibrium
2. energy generation (fusion)
3. mass conservation $dm = 4\pi r^2 \rho dr$
4. radiative transfer (how does energy transport from interior to surface)

$\Downarrow$

Determine $\rho(r)$, $T(r)$ for a star

You will study these equations in detail in 300a & 400a.