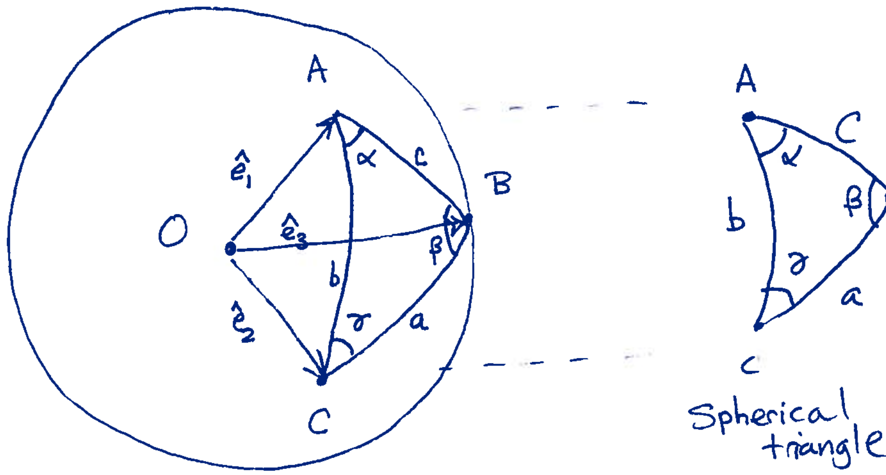
Derivation of Law of Cosines for a spherical triangle

Start with a spherical triangle on a unit sphere



Let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be unit vectors from the center, O, to the points of the triangle A, B, C.

$$|\hat{e}_1| = |\hat{e}_2| = |\hat{e}_3| = 1 \quad \text{for a unit sphere}$$

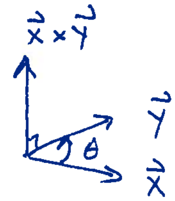
Now, the vectors $\hat{e}_1, \hat{e}_2$ define a plane whose normal vector is given by $\hat{e}_1 \times \hat{e}_2$. Similarly the normals to the other planes formed by the unit vectors are $\hat{e}_1 \times \hat{e}_3$ and $\hat{e}_2 \times \hat{e}_3$.

The angle α in the spherical triangle is the same as the angle between the normal to the planes: $\hat{e}_1 \times \hat{e}_2$ and $\hat{e}_1 \times \hat{e}_3$.

From the defn. of $\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}| \cos\theta$

$$\cos\alpha = \frac{(\hat{e}_1 \times \hat{e}_2) \cdot (\hat{e}_1 \times \hat{e}_3)}{|\hat{e}_1 \times \hat{e}_2|\,|\hat{e}_1 \times \hat{e}_3|}$$

We can simplify the denominator by using the defn. of the cross product



$$|\vec{X} \times \vec{Y}| = |\vec{X}||\vec{Y}| \cdot \sin\theta$$

So
$$|\hat{e}_1 \times \hat{e}_2| = |\hat{e}_1||\hat{e}_2| \cdot \sin b = \sin b$$
$$|\hat{e}_1 \times \hat{e}_3| = |\hat{e}_1||\hat{e}_3| \cdot \sin c = \sin c$$

We now have:

$$\cos\alpha = \frac{(\hat{e}_1 \times \hat{e}_2) \cdot (\hat{e}_1 \times \hat{e}_3)}{\sin b \cdot \sin c}$$

Now, to simplify the numerator, we can use Lagrange's Identity from Vector Analysis:

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

$$\cos\alpha = \frac{(\hat{e}_1 \cdot \hat{e}_1)(\hat{e}_2 \cdot \hat{e}_3) - (\hat{e}_1 \cdot \hat{e}_3)(\hat{e}_2 \cdot \hat{e}_1)}{\sin b \sin c}$$

$$\cos\alpha = \frac{1 \quad \cos a - \cos c \cos b}{\sin b \sin c}$$

$$\Rightarrow \boxed{\cos a = \cos b \cos c + \sin b \sin c \cos\alpha}$$

Law of cosines for a Spherical Triangle

Q.E.D.

Derivation of Law of Sines for a Spherical triangle

USING the same unit sphere from the derivation of the Law of Cosines, we can start with the defn. of the cross product between the normals of the planes defined by $(\hat{e}_1, \hat{e}_2)$ and $(\hat{e}_1, \hat{e}_3)$ to solve for the $\sin\alpha$:

$$\sin\alpha = \frac{|(\hat{e}_1 \times \hat{e}_2) \times (\hat{e}_1 \times \hat{e}_3)|}{|\hat{e}_1 \times \hat{e}_2| \; |\hat{e}_1 \times \hat{e}_3|}$$

The denominator simplies as before :

$$\sin\alpha = \frac{|(\hat{e}_1 \times \hat{e}_2) \times (\hat{e}_1 \times \hat{e}_3)|}{\sin b \quad \sin c}$$

For the numerator, we can use the Vector Quadruple Product formula from Vector Analysis :

$$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = [\vec{A} \cdot (\vec{B} \times \vec{D})]\,\vec{C} - [\vec{A} \cdot (\vec{B} \times \vec{C})]\,\vec{D}$$

So,

$$|(\hat{e}_1 \times \hat{e}_2) \times (\hat{e}_1 \times \hat{e}_3)| = \left| [\hat{e}_1 \cdot (\hat{e}_2 \times \hat{e}_3)]\,\hat{e}_1 - [\cancelto{0}{\hat{e}_1 \cdot (\hat{e}_2 \times \hat{e}_1)}]\,\hat{e}_3 \right|$$

because $\hat{e}_1 \cdot$ vector $\perp$ to $\hat{e}_1$.

$$\sin\alpha = \frac{\hat{e}_1 \cdot (\hat{e}_2 \times \hat{e}_3)}{\sin b \cdot \sin c}$$

Let's calculate the $\sin\beta$ using normals to the planes $(\hat{e}_2 \times \hat{e}_3)$ and $(\hat{e}_1 \times \hat{e}_3)$:

$$\sin\beta = \frac{|(\hat{e}_2 \times \hat{e}_3) \times (\hat{e}_1 \times \hat{e}_3)|}{\sin a \quad \sin c}$$

$$|(\hat{e}_2 \times \hat{e}_3) \times (\hat{e}_1 \times \hat{e}_3)| = \left| \overset{0}{[\hat{e}_2 \cdot (\cancel{\hat{e}_3} \times \hat{e}_3)]}\,\hat{e}_1 - [\hat{e}_2 \cdot (\hat{e}_3 \times \hat{e}_1)]\,\hat{e}_3 \right|$$

$$= \hat{e}_2 \cdot (\hat{e}_3 \times \hat{e}_1)$$

Expressions like these are examples of the Scalar Triple Product:

$$\vec{X} \cdot (\vec{Y} \times \vec{Z}) = \vec{Y} \cdot (\vec{Z} \times \vec{X}) = \vec{Z} \cdot (\vec{X} \times \vec{Y})$$

so $\hat{e}_1 \cdot (\hat{e}_2 \times \hat{e}_3) = \hat{e}_2 \cdot (\hat{e}_3 \times \hat{e}_1) = \hat{e}_3 \cdot (\hat{e}_1 \times \hat{e}_2)$

Therefore $\sin\alpha \cdot \sin b \cancel{\sin c} = \hat{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3) = \sin\beta \sin a \cancel{\sin c}$

$$\Rightarrow \quad \frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b}$$

It is straightforward using the same logic for calculating $\sin\gamma$ that

$$\boxed{\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c}}$$

Q.E.D