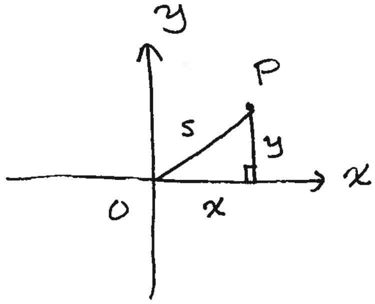
Metrics

A mathematical description for how to measure distances



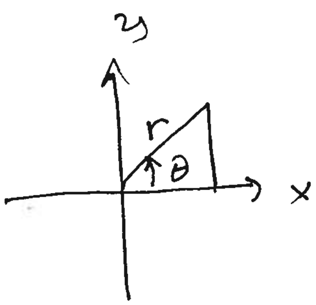
The distance between O and P is $s^2 = x^2 + y^2$ Pythagorean Theorem.

$\mathbb{R}^2$ ← 2 dimension Real space

In the differential limit

$$ds^2 = dx^2 + dy^2$$

coefficients are 1 ⇒ space is flat or Euclidean.



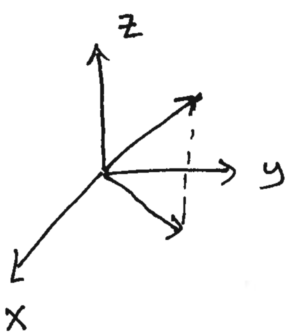
we can transform to polar coordinates

$X = r\cos\theta \qquad y = r\sin y$

$$ds^2 = dr^2 + r^2 d\theta^2$$

coefficients are not 1 but this is still a flat space because a transformation exists to a metric where the coefficients are 1.

Lets extend this concept to $\mathbb{R}^3$:



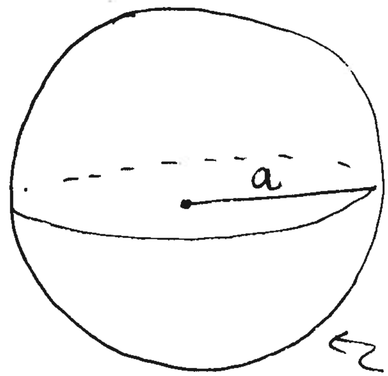
$$ds^2 = dx^2 + dy^2 + dz^2$$

OR in spherical coordinates

$$ds^2 = dr^2 + r^2(d\theta^2 + \cos^2\theta\, d^2\phi)$$

Now consider a $\mathbb{R}^2$ space curved in $\mathbb{R}^3$:

e.g. a sphere



$$ds^2 = a^2(d\theta^2 + \cos^2\theta \, d\phi^2)$$

This is a scale factor

fixed sphere w/ radius a $\Rightarrow dr = 0, \; r = a.$

We need to be careful when talking about curvature in $\mathbb{R}^3$.
There are two types of curvature:

extrinsic = curvature of a lower dimensional space embedded in a higher dimension

intrinsic = curvature that is a fundamental property of the space (N.B. no higher dimensional embedding required!)

~~The scale factor~~

Example of extrinsic curvature:

(1) take a flat sheet of paper ($\mathbb{R}^2$ flat)

(2) roll it into a cylinder

This $\mathbb{R}^2$ space has extrinsic curvature in $\mathbb{R}^3$ but doest not have intrinsic curvature.

The spherical surface in $\mathbb{R}^2$ has intrinsic curvature.

What about space-time?

A flat space-time metric (Special Relativity)

$$ds^2 = \underbrace{dx^2 + dy^2 + dz^2}_{\text{Space-part}} - \underbrace{c^2 dt^2}_{\text{time-part}}$$

time distinct from spatial coordinates.

How does all of this fit together?

Friedmann-Lemaître-Robertson-Walker Metric (FLRW) (General Relativity)

For an isotropic and homogeneous universe (Cosmological Principle)

co-moving coordinates

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \cos^2\theta\, d^2\phi\right)\right]$$

scale factor of expansion (or contraction)

curvature (intrinsic)

$$k = \begin{cases} >0 & \text{Positive (e.g. "spherical")} \\ 0 & \text{Flat} \\ <0 & \text{negative (e.g. "hyperbolic")} \end{cases}$$

$$\frac{d(t)}{d(t_0)} = \frac{\text{co-moving distance measured @ time } t}{\text{co-moving distance measured today}} = \frac{a(t)}{a(t_0)} = \frac{\lambda_{em}}{\lambda_{obs}}$$

Where have we seen this before?

$$1 + z = \frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(t_0)}{a(t)} = \frac{a_0}{a(t)}$$

Don't forget Hubble Law:

$$V = H \cdot d$$

$$V = \dot{d}(t) = \frac{d}{dt}\left(\frac{a(t)}{a_0}\right) d(t_0)$$

$$= \frac{\dot{a}(t)}{a_0} d(t_0)$$

so $$H(t) = \frac{\dot{d}(t)}{d(t)} = \frac{\dot{a}(t)}{a(t)}$$ ~ rate of change of scale factor.

How does $a(t)$ vary w/ cosmological evolution?

General Relativity → Einstein Field Equations → solutions differential equations → $a(t)$

e.g. $$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}$$

Solutions of this equation are beyond what we want to do now.

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example: $E \sim h\nu \sim \frac{hc}{\lambda} \Rightarrow E \propto a^{-1}$

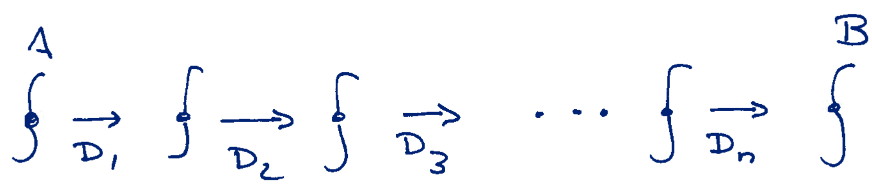
$$n_\gamma = \frac{N_\gamma}{V} \propto a^{-3}$$ if N_γ conserved.

Cosmological Distance Definitions

Remember:
$$\frac{\lambda_{obs}}{\lambda_{em}} = 1+z = \frac{R(t_0)}{R(t)}$$

For $z \ll 1$ we can use $v = c \cdot z$ $\quad D = \dfrac{v}{H(t)} = \dfrac{cz}{H(t)}$

Imagine two galaxies A and B separated by a large distance ($z \not\ll 1$). In order to measure the co-moving radial distance, we need a chain of observers



each observer in a galaxy whose distance is small such that $v \sim c \cdot z$ is appropriate. They must all measure $D_1, D_2, \ldots D_n$ at the <u>same</u> proper time to determine

$$D_{now} = D_1 + D_2 + \cdots D_n$$

<u>This is impractical</u> !

Therefore, alternative distance definitions have been devised ...

1. Co-moving radial distance $\quad D_{now} \leftarrow$ what we have been using to date

2. Angular diameter distance $\quad D_A \equiv \dfrac{\text{size of object}}{\text{angular extent of object in the sky}} = \dfrac{R}{\theta}$

(Problem: Need to know size of object!)

3. Luminosity Distance $\quad F = \dfrac{L}{4\pi D_L^2}$

(Problem: Need to know L) * useful for "standard candles"!

4. Light travel time distance $\quad D_{ltt} = c \cdot (t_0 - t_{em})$

t_0 = age of universe

Event Horizon is $D_{ltt} = c t_0$

(Need to know t_0 and figure out t_{em})

It turns out D_A & D_L are related:

$$L = 4\pi R^2 \sigma T_{em}^4 \quad \text{for a blackbody at redshift } z$$

$$T_{obs} = \frac{T_{em}}{1+z}$$ ← you will prove this in Hmrk # 8

$$F_{obs} = \theta^2 \cdot \sigma T_{obs}^4$$

$$D_L^2 = \frac{L}{4\pi F_{obs}} = \frac{\cancel{4}\cancel{\pi} R^2 \cancel{\sigma} T_{em}^4}{\cancel{4}\cancel{\pi} \theta^2 \cancel{\sigma} T_{obs}^4}$$

remember $D_A = \frac{R}{\theta}$

$$D_L^2 = D_A^2 \cdot \frac{T_{em}^4}{T_{obs}^4} = D_A^2 \cdot (1+z)^4$$

$$\boxed{D_L = D_A \cdot (1+z)^2}$$