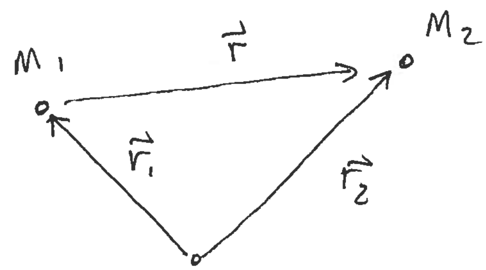
Kepler's Laws



$$\vec{r} = \vec{r}_2 - \vec{r}_1$$

$$\vec{F}_{grav} = -\frac{G m_1 m_2}{r^2} \underbrace{\frac{\vec{r}}{r}}_{\substack{\text{unit vector} \\ \text{in } \vec{r} \text{ direction}}} = -\frac{G m_1 m_2}{r^3} \vec{r}$$

also $\vec{F} = m\vec{a}$ Newton's 2nd Law

Eqn of motion of m_2: $\cancel{m_2} \vec{a}_2 = -G m_1 \cancel{m_2} \frac{\vec{r}}{r^3}$

Eqn. of motion of m_1: $-\ \cancel{m_1} \vec{a}_1 = +G \cancel{m_1} m_2 \frac{\vec{r}}{r^3}$

$$\vec{a}_2 - \vec{a}_1 = -G(m_1 + m_2)\frac{\vec{r}}{r^3}$$

$$\vec{a} = -\frac{G(m_1+m_2)}{r^3}\vec{r}$$

$$\frac{d^2\vec{r}}{dt^2} = -\frac{G(m_1+m_2)}{r^3}\vec{r}$$

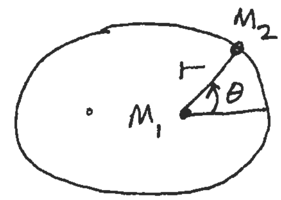
This is a second order vector differential equation

Solution gives $\vec{r}(t)$

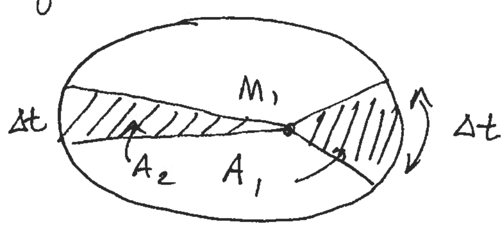
Need 6 constants to completely specify motion of a mass.

Keplers 1st Law: The orbit of a planet is an ellipse, one focus of which is in the sun

$$r(\theta) = \frac{P}{1 + e\cos\theta}$$



Kepler's 2ND Law: The radius vector of a planet sweep equal areas in equal amounts of time



$A_1 = A_2$ in same Δt

⇒ Planet moves faster when closer to M_1

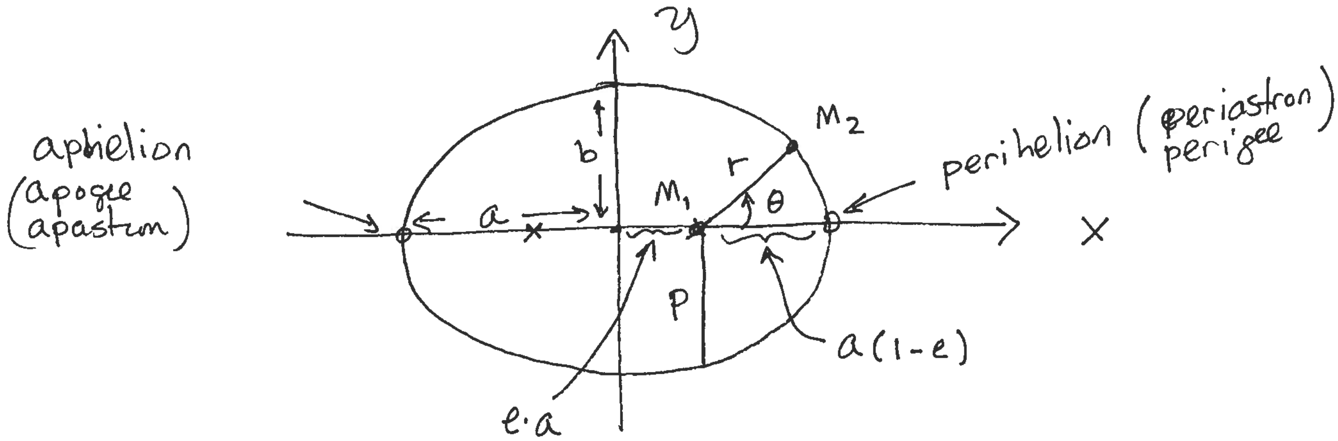
$$V(r) = \sqrt{G(M_1+M_2)\left(\frac{2}{r} - \frac{1}{a}\right)} \quad \text{for elliptical orbit}$$

Kepler's 3rd Law: The ratio of the cubes of the semi-major axes of the orbits of two planets = the ratio of the squares of their orbital periods.

$$\frac{P_1^2}{P_2^2} = \frac{a_1^3}{a_2^3}$$

The exact form from Newton's Laws gives: $$P^2 = \frac{4\pi^2}{G(M_1+M_2)}\, a^3$$

Kepler's 1st Law
Properties of Ellipses



$A = \pi a b$ = area of an ellipse

Cartesian Coordinates: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

a ← semi-major axis; b ← semi-minor axis

$$b = a\sqrt{1 - e^2}$$

e ← eccentricity

$$\Rightarrow \quad e = \sqrt{1 - \frac{b^2}{a^2}} \qquad e \in [0, 1)$$

Polar Coordinates

$$r(\theta) = \frac{P}{1 + e\cos\theta}$$

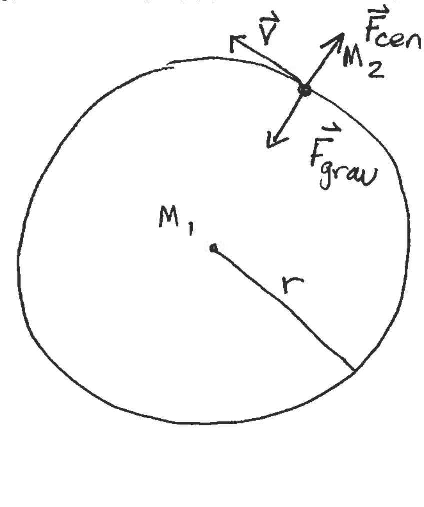
P ← semi-latus rectum $= a(1-e^2)$

e ← eccentricity; θ ← "true anomaly"

In General, $r(\theta)$ is the form of all conic sections

$e = 0$	circle
$e < 1$	ellipse
$e = 1$	parabola
$e > 1$	hyperbola

Circular Velocities



$$\vec{F}_{cen} = \vec{F}_{grav}$$

$$\frac{\cancel{m_2} v_c^2}{r} = \frac{G M_1 \cancel{m_2}}{r^2}$$

$$V_c = \sqrt{\frac{GM}{r}} \sim \frac{1}{\sqrt{r}}$$

For the Earth $e \simeq 0.0167 \Rightarrow$ very close to circular!

<u>Example</u>: Earth velocity around the Sun:

$$V_c = \left(\frac{6.67 \times 10^{-8}\ cm^{\cancel{3}\,2} \cdot s^{-2} \cdot \cancel{g}^{-1} \times 2 \times 10^{33}\ \cancel{g}}{149.5 \times 10^{11}\ \cancel{cm}} \right)^{1/2} \sim 30\ km/s$$

For a planet 2x farther away, what is the velocity?

$$V_{planet}\,(2AU) = 30\ km/s \cdot \frac{1}{\sqrt{2}} \sim 21\ km/s$$

For a planet 4x farther away:

$$V_{planet}\,(4AU) = 30\ km/s \cdot \frac{1}{2} \sim 15\ km/s \qquad \text{etc.}$$

IN Reality, planets move in <u>elliptical</u> <u>orbits</u>

$$V(r) = \sqrt{G(m_1 + m_2)\left(\frac{2}{r} - \frac{1}{a}\right)}$$

If $m_2 \ll M_1 \Rightarrow V(r) = \sqrt{GM_1\left(\frac{2}{r} - \frac{1}{a}\right)}$

Note that $v(r)$ still $\sim \frac{1}{\sqrt{r}}$

When $a = r$ we have a circular orbit and

$$v(r) = \sqrt{GM\left(\frac{2}{r} - \frac{1}{r}\right)} = \sqrt{\frac{GM}{r}}$$ same as for circular velocity ✓

What happens when $a \to \infty$?

$$v(r) = \sqrt{\frac{2GM}{r}} = \sqrt{2}\, v_{circular}$$

Where have we seen this before? It is the escape velocity.

$$K.E. \geq P.E.$$

$$\frac{1}{2}\cancel{m} v_{esc}^2 \geq \frac{GM\cancel{m}}{r}$$

$$\Rightarrow v_{esc} \geq \sqrt{\frac{2GM}{r}}$$ escape velocity

For the Earth orbiting @ 1 AU:

$$v_{esc} = \sqrt{2} \cdot 30 \text{ km/s} \sim 42 \text{ km/s}$$

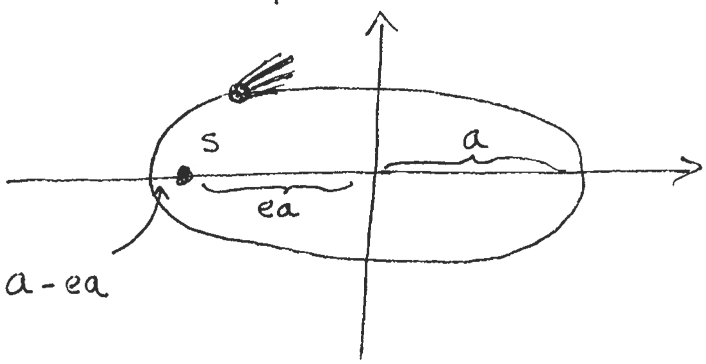
SO an object @ 1 AU moving with $v \gtrsim 42$ km/s is unbound and will escape from the solar system (escape from the gravity of the Sun).

These "orbits" correspond to parabolic ($v = v_{esc}$) and hyperbolic ($v > v_{esc}$) orbits.

EXAMPLE: HALLEY'S COMET

$a = 17.8$ AU $\qquad e = 0.967$

What is the perihelion and aphelion distance?



aphelion $= r_a = a + ea = a(1+e)$

$= 17.8 \text{ AU } (1+0.967)$

$= 35.0$ AU

perihelion $= r_p = a - ea = a(1-e)$

$= 17.8 \text{ AU } (1-0.967)$

$= 0.587$ AU

What is the speed of Hally's comet when it passes the Earth ($r = 1$ AU)?

$$V(r) = \sqrt{GM_\odot\left(\frac{2}{r} - \frac{1}{a}\right)} = \sqrt{GM_\odot\left(\frac{2}{1\text{ AU}} - \frac{1}{17.8\text{ AU}}\right)}$$

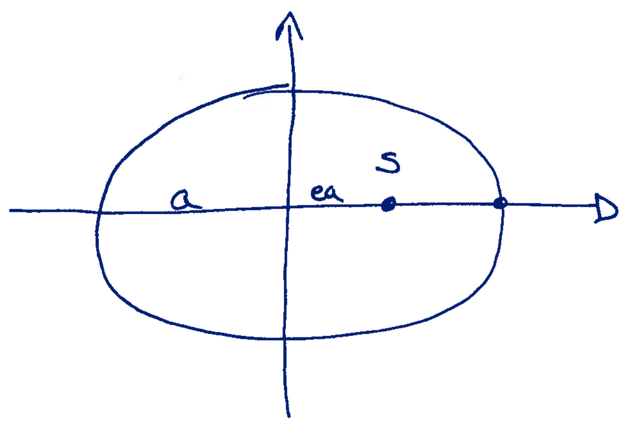
$1 \text{ AU} = 1.496 \times 10^{13}$ cm

$V(1 \text{ AU}) = 41.6$ km/s ← just less than $V_{esc}(1 \text{ AU})$! ~ 42 km/s

Note that e is close to 1, so Halley's comet is close to unbound!

Pluto's Orbit

$a = 39.482 \text{ AU}$ $\qquad$ $e = 0.249$



$r_p = a - ea = a(1-e) = 39.482\,(1-0.249) = 29.65 \text{ AU}$

$r_a = a + ea = a(1+e) = 49.31 \text{ AU}$

$P = (39.482 \text{ AU})^{3/2} = 248.1 \text{ years}$

Halley's Comet

$a_{\text{Halleys}} = (75.3 \text{ years})^{2/3} = 17.8 \text{ AU}$

Kepler's Law Con't.

Velocity example:

Is it easier to send a rocket directly from the Earth to the Sun or for the rocket to escape from the solar system?

$$V_{esc} = \sqrt{\frac{2GM}{r}}$$

To Escape you need to ① Launch from Earth $\Rightarrow \Delta V = V_{esc}^{\oplus} \sim 11$ km/s

(2.) Escape from Solar System $\Rightarrow \Delta V = V_{esc}(1AU) \sim 42$ km/s

BUT the Earth is already moving @ 30 km/s $\Rightarrow \Delta V = 42$ km/s $- 30$ km/s $= 12$ km/s

$$\Delta V_{TOT} = 11 + 12 = 23 \text{ km/s}$$

To directly fly to Sun (1.) Launch from Earth $\Delta V \sim 11$ km/s

(2.) Loose 30 km/s rotation speed

$$\Delta V_{TOT} = 30 + 11 = 42 \text{ km/s}$$

$\Rightarrow$ almost twice as much ΔV needed!

$$M_{\oplus} = 5.97 \times 10^{27} \text{ g}$$

$$R_{\oplus} = 6371 \text{ km} = 6.4 \times 10^{8} \text{ cm}$$

Kepler's 3rd Law

$$P^2 = \frac{4\pi^2}{G(M_1+M_2)}\, a^3$$

A word about units :

$$G = 6.67 \times 10^{-8}\ cm^3 \cdot s^{-2} \cdot g^{-1}$$

what if we convert to $AU^3\ yrs^{-2} \cdot M_\odot^{-1}$?

$$G = 6.67\times10^{-8}\ cm^3 \cdot s^{-2} \cdot g^{-1} \times \left(\frac{1\ AU}{1.496\times10^{13}\ cm}\right)^3 \times \left(\frac{365.25\times24\times3600\ s}{1\ yr}\right)^2$$

$$\times \left(\frac{\cancel{2}\ 1.989 \times 10^{33}\ g}{1\ M_\odot}\right)$$

$$G = 39.48 = 4\pi^2 \ !$$

This means Kepler's 3^{rd} Law for $M_2 \ll M_1$ and $M_1 = 1 M_\odot$ becomes :

$$\boxed{P^2_{(yr)} = 1 \cdot a^3_{(AU)}}$$ Solar System

Note : in these units $(AU, yrs, M_\odot)$ ~~the~~ Kepler's Second Law

$$v(r) = \sqrt{4\pi^2\left(\frac{2}{r} - \frac{1}{a}\right)}$$ where r and a are in AU !

Examples Kepler's 3^{rd} Law

1.) Determining the Period of an orbit

Halley's Comet $\quad a = 17.8$ AU

$$P^2_{Halleys} = a^3_{Halleys}$$

$$P_{Halley} = (17.8)^{3/2} \simeq 75 \text{ years}$$

2.) Determining the semi-major axis:

Mars $\quad P_{\male} = 686.97$ days $= 1.88$ years

$$a^3_{\male} = P^2_{\male}$$

$$a_{\male} = (1.88)^{2/3} = 1.52 \text{ AU}$$

3.) Determining the mass of Mars using its tiny moon

Phobos: $\quad P = 0.3189$ days $\quad a = 9370$ km

7.6536 hours

$$m_{phobos} \ll M_{\male} \Rightarrow P^2 = \frac{4\pi^2}{GM_{\male}} a^3$$

$$M_{\male} = \frac{4\pi^2}{G}\frac{a^3}{P^2}$$

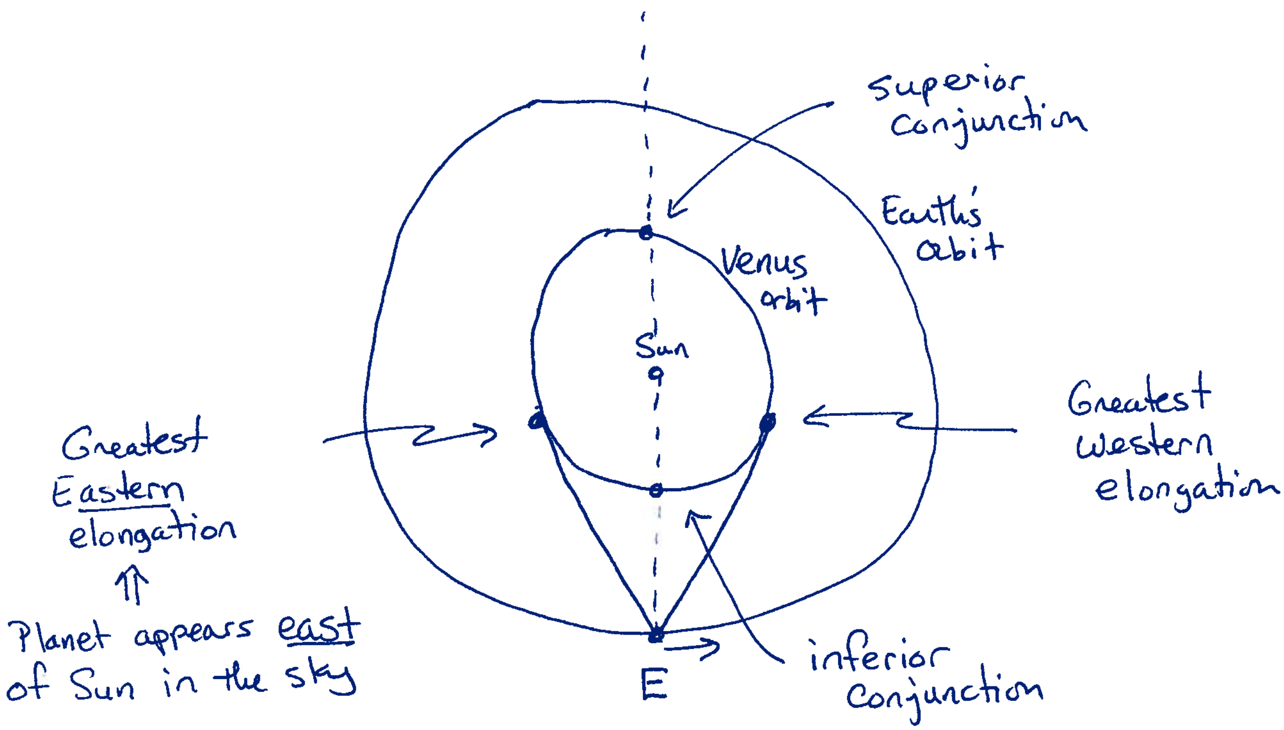
$$M_{\male} = \frac{4\pi^2}{6.67\times10^{-8}\ cm^3 \cdot s^{-2} \cdot g^{-1}} \times \frac{(9370\times10^5 cm)^3}{(0.3189\times24\times3600\,s)^2} = 6.41\times10^{26}\ g$$

$$M_{\oplus} = 5.97\times10^{27}\ g \Rightarrow M_{\male} = 0.107\ M_{\oplus}$$

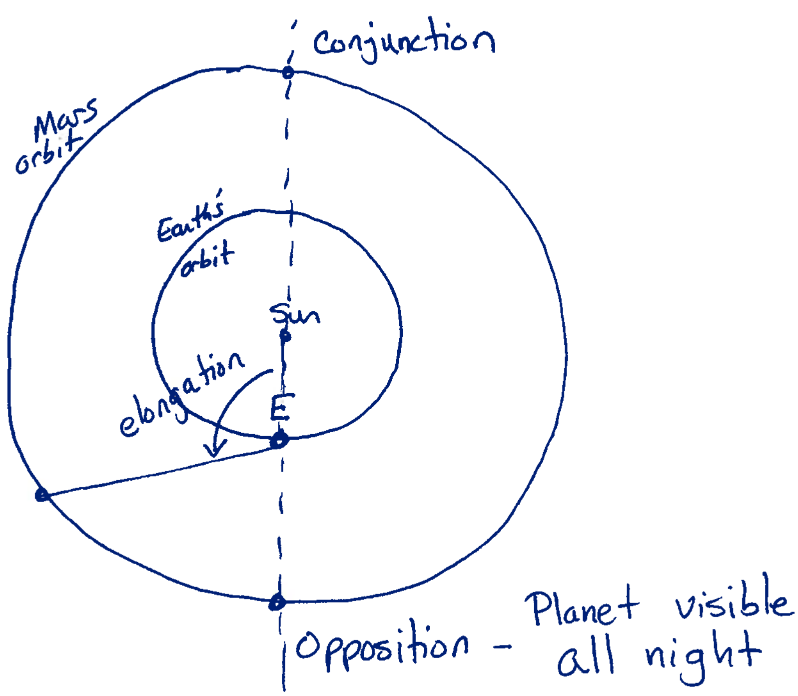
Planetary Configurations

Basic Terminology

Inferior Planets



Superior Planets



Synodic vs. Siderial Period

S = Synodic Period = time for an object to reappear in the same point in the sky relative to the Sun, as observed from the Earth.

P = Siderial Period = time to make 1 full orbit

Example: Moon has a siderial Period $P_{moon} = 27.3$ days

$P_{\oplus} = 365.25$ days

Each day, the Moon gains on the Sun by an angle of $\left(\frac{360^\circ}{P_{moon}} - \frac{360^\circ}{P_{\oplus}}\right)$

In one Synodic Period, this "gain" must equal 360°.

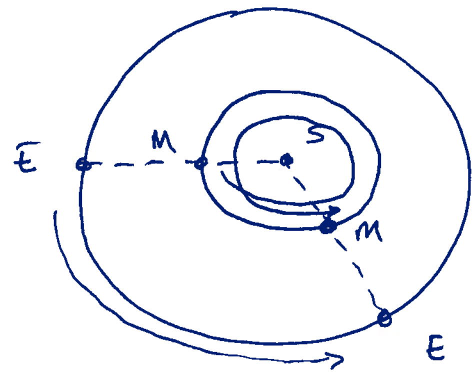
$$\Rightarrow \quad \frac{360^\circ}{S} = \frac{360^\circ}{P_{moon}} - \frac{360^\circ}{P_{\oplus}}$$

$$\Rightarrow \quad \frac{1}{S} = \frac{1}{P_{moon}} - \frac{1}{P_{\oplus}}$$

$S_{moon} = 29.5$ days for phases to repeat!

Synodic Period vs. Sidereal Period

Inferior Planet:



$$\left(\frac{360^\circ}{P_\oplus}\right)\cdot S + 360^\circ = \left(\frac{360^\circ}{P_{mercury}}\right)\cdot S$$

angular distance covered by Earth

$$\frac{1}{P_{mer}} = \frac{1}{P_\oplus} + \frac{1}{S}$$

or $$\frac{1}{S} = \frac{1}{P_{mer}} - \frac{1}{P_\oplus}$$

Exterior Planet

$$\left(\frac{360^\circ}{P_{Jup}}\right)\cdot S + 360^\circ = \left(\frac{360^\circ}{P_\oplus}\right)\cdot S$$

$$\frac{1}{P_{Jup}} = \frac{1}{P_\oplus} - \frac{1}{S}$$

$$\frac{1}{S} = \frac{1}{P_\oplus} - \frac{1}{P_{Jup}}.$$

Reality ... Systems of Multiple bodies

Two-body problem : $$\frac{d^2\vec{r}}{dt^2} = -G(m_1+m_2)\frac{\vec{r}}{r^3}$$

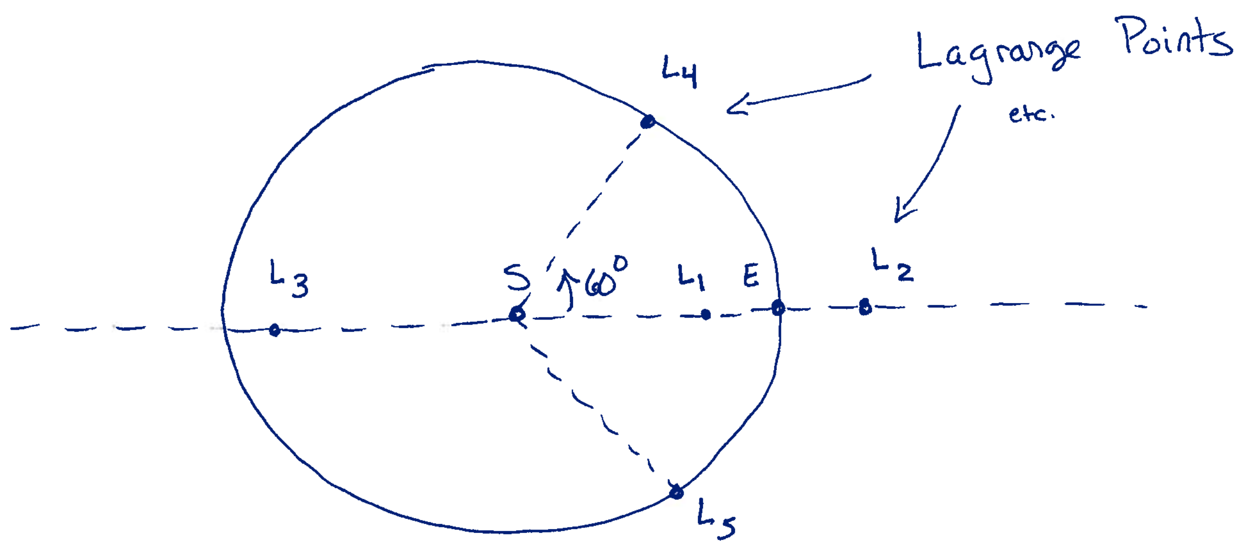
Generalizes to : $$\frac{d^2\vec{r}_j}{dt^2} = \sum_{\substack{i=1 \\ i\neq j}}^{n} G m_i \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3}$$

Lagrange Points - for a 3rd body whose mass is negligable

$m_3 \ll m_2$ or m_1

||

Points where gravitational forces balance



L_1 L_2 L_3 unstable equilibrium

L_4 L_5 stable equilibrium

examples : WMAP
JWST placed @ L_2 !

Trojan asteroids orbit w/ Jupiter @ L_4 & L_5 !

Derivation of Kepler's 1st Law

$$\vec{a} = -\frac{GM}{r^2}\hat{r} = -\frac{GM}{r^3}\vec{r}$$

"Specific" Angular Momentum is conserved $\Rightarrow \quad \vec{r}\times\vec{v} \equiv \vec{h} = \text{const} \quad \Rightarrow \frac{d\vec{h}}{dt} = 0$

$$\vec{h} = \vec{r}\times\vec{v} = r\hat{r}\times\frac{d}{dt}(r\hat{r})$$

$$= r\hat{r}\times\left(r\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r}\right)$$

vector identity: $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$

$$\vec{h} = r^2\left(\hat{r}\times\frac{d\hat{r}}{dt}\right) + r\frac{dr}{dt}\overset{0}{\cancel{(\hat{r}\times\hat{r})}}$$

Now form the quantity $\vec{a}\times\vec{h}$

$$\vec{a}\times\vec{h} = \left(-\frac{GM}{\cancel{r^2}}\hat{r}\right) \times \left(\cancel{r^2}\hat{r}\times\cancel{r^2}\frac{d\hat{r}}{dt}\right)$$

$$= -GM\hat{r}\times\left(\hat{r}\times\frac{d\hat{r}}{dt}\right)$$

vector identity: $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$$= -GM\left[\hat{r}\underbrace{\overset{0}{\cancel{\left(\hat{r}\cdot\frac{d\hat{r}}{dt}\right)}}}_{\text{they are }\perp\,!} - \frac{d\hat{r}}{dt}\overset{1}{\cancel{(\hat{r}\cdot\hat{r})}}\right]$$

because $\frac{d\vec{h}}{dt} = 0$!

$\vec{a}\times\vec{h} = GM\frac{d\hat{r}}{dt} \qquad$ Now $\quad \vec{a}\times\vec{h} = \frac{d\vec{v}}{dt}\times\vec{h} = \frac{d}{dt}(\vec{v}\times\vec{h})$

So $\quad \frac{d}{dt}(\vec{v}\times\vec{h}) = GM\frac{d\hat{r}}{dt}$

Integrating:

$$\int d(\vec{v}\times\vec{h}) = GM\int d\hat{r}$$

$$\Rightarrow \quad \vec{v}\times\vec{h} = GM\hat{r} + \vec{C}$$

constant vector but $\vec{C}$ must lie in same plane as $\hat{r}$.

Now take:

$$\vec{r} \cdot (\vec{v} \times \vec{h}) = \vec{r} \cdot GM\hat{r} + \vec{r} \cdot \vec{c}$$

If θ is angle between $\vec{r}$ and $\vec{c}$ then $\vec{r} \cdot \vec{c} = rc\cos\theta$

$$\vec{r} \cdot (\vec{v} \times \vec{h}) = GMr(\underbrace{\hat{r} \cdot \hat{r}}_{1}) + rc\cos\theta$$

Solving for r:

$$r = \frac{\vec{r} \cdot (\vec{v} \times \vec{h})}{GM + c\cos\theta}$$

Define $e = c/GM$

$$r = \frac{1}{GM} \frac{\vec{r} \cdot (\vec{v} \times \vec{h})}{1 + e\cos\theta}$$

Now $\vec{r} \cdot (\vec{v} \times \vec{h}) = (\vec{r} \times \vec{v}) \cdot \vec{h} = \vec{h} \cdot \vec{h} = h^2$

So $r = \frac{1}{GM} \frac{h^2}{1 + e\cos\theta} = \frac{eh^2/c}{1 + e\cos\theta}$ ← this is the polar equation for a conic section!

$\Rightarrow$ For a closed orbit, $r(\theta)$ is described by an ellipse.