# Electrostatic Bremsstrahlung – "Free-Free" Radiation

ASTR 300B

In plasma, $e^-$ and ions are accelerated as they pass each other on scales < Debye length of plasma. $D_e \approx \left(\frac{kT}{4\pi n_e e^2}\right)^{1/2}$ see Jackson Classical Electrodynamics

The Power radiated from an accelerated charge is given by classical (non-relativistic) Larmor's formula:

cgs: $P(t) = \frac{2}{3}\frac{e^2}{c^3}|\vec{a}(t)|^2 \ \mathrm{erg \cdot s^{-1}}$ { N.B. $1\,e \simeq 4.8\times10^{-10}$ statcoloumb

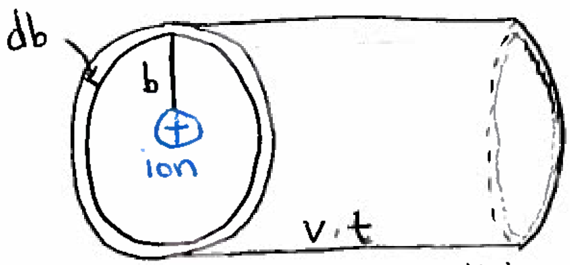
SINCE $a = \frac{F}{m} \Rightarrow P \sim a^2 \sim \frac{1}{m^2} \Rightarrow \frac{P_{e^-}}{P_{ions}} \sim \left(\frac{m_p}{m_e}\right)^2 \sim 4\times10^6 \Rightarrow$ ONLY care about radiation from $e^-$ !!

$e^- - e^-$ interactions negligible because E fields opposite in magnitude $\Rightarrow$ no net radiated Power at large d $\Rightarrow$ only $e^-$-ion important !!

How do we figure out $j_\nu$ ?

1. Fourier Transform $P(t) \rightarrow P(\omega)$ $\quad \omega = 2\pi\nu$ angular frequency
   $\mathrm{erg \cdot s^{-1}}$ $\quad \mathrm{erg \cdot Hz^{-1}}$ Note this will be function of b and v too.

2. For single ion, calculate rate of $e^-$ (per s) passing ion at impact parameters $b + db$ with speed $v + dv$




units think:

$\frac{1}{t} = n\sigma v$

distribution of $e^-$ speeds
$\int f(v)\,dv = 1$ Normalized

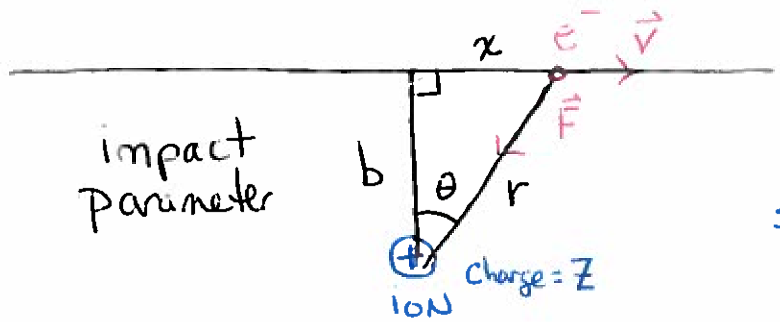
$n_e \cdot (2\pi b\,db) \cdot v\,f(v)\,dv$
$\mathrm{cm^{-3}} \quad \mathrm{cm^2} \quad \mathrm{cm \cdot s^{-1}}$ ✓

3. Multiply by $n_{ions} = n_i$ and $\int_b \int_v$ to get total rate

$$j_\nu = \frac{1}{4\pi}\int_v \int_b P(\omega; v, b)\, n_i n_e\, (2\pi b\,db)\, v\, f(v)\,dv$$

isotropic $\rightarrow \mathrm{Ster^{-1}}$ $\quad \mathrm{erg \cdot Hz^{-1}}\ \mathrm{cm^{-3}}\ \mathrm{cm^{-3}}\ \mathrm{cm^{2}}\ \mathrm{cm \cdot s^{-1}} = [j_\nu]$ ✓

Let's assume deflection angle is small such that we can approximate $e^-$ as flying in straight line




$$x \simeq v \cdot t$$
$$r^2 = b^2 + x^2$$
$$\sin\theta = \frac{vt}{r}$$
$$\cos\theta = \frac{b}{r}$$

Let's decompose $\vec{a}$ into $a_{\parallel}$ ($x$ direction) and $a_{\perp}$

$$a_{\parallel} = \frac{F_{\parallel}}{m_e} = \frac{-e^2 \cdot Z}{m_e r^2} \sin\theta \qquad a_{\perp} = \frac{F_{\perp}}{m_e} = \frac{e^2 \cdot Z}{m_e r^2} \cos\theta$$

$$a_{\parallel} = \frac{-Ze^2}{m_e r^2} \frac{vt}{r} \qquad a_{\perp} = \frac{Ze^2}{m_e r^2} \cdot \frac{b}{r}$$

$$a_{\parallel} = -\frac{Ze^2 \cdot vt}{m_e (b^2+v^2t^2)^{3/2}} \qquad a_{\perp} = \frac{Ze^2 \cdot b}{m_e (b^2+v^2t^2)^{3/2}}$$

Now to calculate $P(\omega)$ we need Fourier transform of $\vec{a}(t)$

$$\hat{a}_{\parallel}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a_{\parallel}(t)\, e^{i\omega t}\, dt \qquad \hat{a}_{\perp} = \frac{1}{2\pi}\int_{-\infty}^{\infty} a_{\perp}(t)\, e^{i\omega t}\, dt$$

$$\hat{a}_{\parallel}(\omega) = \frac{Ze^2 v}{2\pi m_e}\int_{-\infty}^{\infty} \frac{t \cdot e^{i\omega t}}{(b^2+v^2t^2)^{3/2}}\, dt \qquad \hat{a}_{\perp} = \frac{Ze^2 b}{2\pi m_e}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(b^2+v^2t^2)^{3/2}}\, dt$$

$$\hat{a}_{\parallel}(\omega) = \frac{Ze^2}{2\pi m_e bv} \cdot \frac{2\omega b}{v}\, i\, K_0\left(\frac{\omega b}{v}\right) \qquad \hat{a}_{\perp}(\omega) = \frac{Ze^2}{2\pi m_e bv} \cdot \frac{2\omega b}{v} K_1\left(\frac{\omega b}{v}\right)$$

$K_0(x)$ and $K_1(x)$ are modified Bessel Functions of $2^{nd}$ Kind

Asymptotic Properties: $K_0(x) \rightarrow -\ln x \quad x \ll 1$ (Low $\nu$)
$K_1(x) \rightarrow 1/x \quad x \ll 1$

$K_0(x)$ and $K_1(x) \rightarrow \sqrt{\frac{\pi}{2x}}\, e^{-x} \quad x \gg 1$ (high $\nu$)

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$$j_\nu = \frac{1}{4\pi} \int_{b=b_{min}}^{b_{max}} \int_{v=v_{min}}^{v_{max}} P(\omega; v, b)\, n_i n_e\, 2\pi b\, db\, v\, f(v)\, dv$$

ster$^{-1}$ (isotropic)

erg $\cdot$ Hz$^{-1}$ $\cancel{cm}^{-3}$ $\cancel{cm}^{3}$ $\cancel{cm}^{2}$ $\cancel{cm} \cdot s^{-1}$ = [$j_\nu$] ✓

$$P(\omega; v, b) \sim \frac{e^2}{c^3} |\hat{a}(\omega)|^2 \sim \frac{e^2}{c^3} \left[ |\hat{a}_{\parallel}(\omega)|^2 + |\hat{a}_{\perp}(\omega)|^2 \right]$$

Since $P(\omega)$ is nearly flat and doesn't fall of exponentially until high $\nu$, we can ignore $\hat{a}_{\parallel}(\omega)$ and just use $\hat{a}_{\perp}$ in low $\nu$ limit:

$$P(\omega; v, b) \sim \frac{e^2}{c^3} \cdot \left( \frac{Ze^2}{m_e b v} \cdot \frac{\cancel{\omega b}}{\cancel{v}} \cdot \frac{\cancel{v}}{\cancel{\omega b}} \right)^2 \sim \frac{Z^2 e^6}{c^3 m_e^2 b^2 v^2} \sim \frac{1}{b^2 v^2}$$

Let's $\int db$ first. This integral logarithmically diverges, so we need limits $b_{min}$ and $b_{max}$ (which could be functions of $v$)

$$j_\nu \sim \int_v \int_{b=b_{min}(v)}^{b_{max}(v)} \frac{Z^2 e^6 n_i n_e}{c^3 m_e^2 b^{\cancel{2}} v^2} \cdot \cancel{b}\, db\, \cancel{v}\, f(v)\, dv$$

$$\sim \int_v \frac{Z^2 e^6 n_i n_e}{c^3 m_e^2} \ln\left(\frac{b_{max}(v)}{b_{min}(v)}\right) \cdot \frac{f(v)}{v} dv$$

In LTE $f(v)$ = Maxwell Boltzman $= 4\pi \left(\frac{m_e}{2\pi kT}\right)^{3/2} v^2 e^{-m_e v^2/2kT}$

For a lower limit, photons are quantized $\Rightarrow E_\gamma = h\nu = \frac{1}{2} m_e v_{min}^2$

$$j_\nu \sim \int_{v_{min}}^{v_{max} \to \approx \infty} \frac{Z^2 e^6 n_i n_e}{c^3 m_e^2} \ln\left(\frac{b_{max}(v)}{b_{min}}\right) \cdot \left(\frac{m_e}{kT}\right)^{3/2} \cdot v \cdot e^{-m_e v^2/2kT} dv$$

Note that $\int x e^{-ax^2} dx = \frac{-e^{-ax^2}}{2a}$ $\quad a = \frac{m_e}{2kT}$

$$j_\nu \sim \frac{Z^2 e^6 n_i n_e}{c^3 m_e^2} \cdot \left(\frac{m_e}{kT}\right)^{3/2} \cdot \left(\frac{kT}{m_e}\right) \cdot \bar{g}_{ff} \cdot e^{-h\nu/kT}$$

$$j_\nu = \frac{8}{3}\left(\frac{2\pi}{3}\right)^{1/2} \frac{Z^2 e^6}{m_e^2 c^3} \left(\frac{m_e}{kT}\right)^{1/2} n_e n_i e^{-h\nu/kT} \overline{g_{ff}}$$

$erg \cdot s^{-1} \cdot cm^{-3} \cdot ster^{-1} \cdot Hz^{-1}$

"Gaunt Factor" (averaged over $\nu$)

At low frequencies the gaunt factor is

$$\overline{g_{ff}} \simeq \frac{\sqrt{3}}{\pi}\left[\ln \frac{(2kT)^{3/2}}{\pi Z e^2 m_e^{1/2} \nu} - \frac{5\gamma}{2}\right]$$

Euler's const

In General - need to consider quantum effects

$$\simeq 6.155 \left(Z\, \nu_{GHz}\right)^{-0.118} \left(T/10^4 K\right)^{0.177}$$

valid $\nu \ll \frac{kT}{h}$

Why does $\overline{g_{ff}} \sim \ln\left(\frac{1}{\nu}\right)$? Let's explore classical limits of $b_{min}$ and $b_{max}$.

Since $P(\omega)$ exponentially decays at $\frac{\omega b}{v} \Rightarrow b_{max} \approx \frac{v}{\omega} = \frac{v}{2\pi\nu}$

For $b_{min}$, let's analyze the net momentum impulse:

$$\Delta p_e = \int_{-\infty}^{\infty} F_\perp \, dt = \int_{-\infty}^{\infty} \frac{Ze^2 \cos\theta}{r^2} dt$$

Let's change $\int dt \to \int d\theta$: $\cos\theta = \frac{b}{r} \Rightarrow r^2 = \frac{b^2}{\cos^2\theta}$ $\quad \tan\theta = \frac{x}{b} = \frac{vt}{b}$

$$dt = \frac{b}{v} \sec^2\theta \, d\theta \quad \Leftarrow \quad d(\tan\theta) = \sec^2\theta \, d\theta = \frac{v}{b} dt$$

$$\Delta p_e = \frac{Ze^2}{b^2} \int_{-\pi/2}^{\pi/2} \cos^3\theta \cdot \frac{b}{v} \sec^2\theta \, d\theta = \frac{Ze^2}{bv} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = \frac{2Ze^2}{bv}$$

The maximum possible $\Delta p_e = 2 m_e v = \frac{2Ze^2}{b_{min} v}$

$$\Rightarrow b_{min} \approx \frac{Ze^2}{m_e v^2}$$

So $\ln\left(\frac{b_{max}}{b_{min}}\right) \approx \ln\left(\frac{v}{2\pi\nu} \cdot \frac{m_e v^2}{Ze^2}\right) \sim \ln\left(\frac{1}{\nu}\right)$ !

Very weak dependence on $\nu$ at low $\nu$.