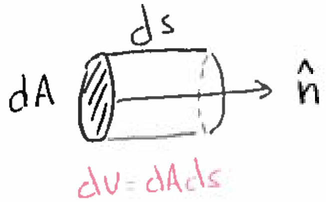

Consider I_ν passing through dA in the normal direction $\hat{n}$.

By definition $\quad I_\nu = \dfrac{dE}{dA\, d\Omega\, dt\, d\nu}$

so
$$\frac{d}{ds}(I_\nu) = \frac{dE}{\underbrace{dA\,ds}_{dV}\, d\Omega\, dt\, d\nu} = \frac{dE}{dV\, d\Omega\, dt\, d\nu}$$

The total change in I_ν along ds must be due to emission, absorption, and/or scattering along ds direction.

We define two new quantities

j_ν = Emissivity Coefficient = rate of energy emitted per Hz at frequency ν ~~into~~ per solid angle $d\Omega$ per unit volume

$$j_\nu = \frac{dE}{dV\, d\Omega\, dt\, d\nu} \qquad erg \cdot cm^{-3} \cdot s^{-1} \cdot ster^{-1} \cdot Hz^{-1}$$

$\Rightarrow \quad \dfrac{dI_\nu}{ds} = +\, j_\nu \quad$ is intensity added to beam

α_ν = Absorption Coefficient = proportional to number of photons absorbed along line of sight ds per cm^{-1} $\qquad cm^{-1}$

<u>Define</u> $\quad \dfrac{dI_\nu}{ds} = -\,\alpha_\nu I_\nu \quad$ is intensity taken out of beam

1D Radiative Transfer Equation:

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$$

$$erg \cdot s^{-1} \cdot \underline{cm^{-3}} \cdot ster^{-1} \cdot Hz^{-1}$$

If we divide both sides by α_ν, we define 2 new quantities:

$$\frac{dI_\nu}{\alpha_\nu ds} = \frac{j_\nu}{\alpha_\nu} - I_\nu \implies \frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

optical depth

Source Function

① Optical Depth : τ

$$d\tau_\nu \equiv \alpha_\nu ds$$

$$\tau_\nu = \int_0^L \alpha_\nu(s)\, ds$$

$cm^{-1} \cdot cm$ = UNITLESS

total path length

Optical Depth is unitless

② SOURCE FUNCTION : S_ν

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu}$$

Same units as I_ν!

$erg \cdot s^{-1} \cdot cm^{-2} \cdot ster^{-1} \cdot Hz^{-1}$

Consider the case of thermodynamic equilibrium:

Now $I_\nu = B_\nu(T)$ Planck Function

Thermodynamic equilibrium $\Rightarrow \frac{dI_\nu}{ds} = 0 \Rightarrow I_\nu = B_\nu(T) = $ CONST

$$0 = j_\nu - \alpha_\nu B_\nu(T)$$

$$B_\nu(T) = \frac{j_\nu}{\alpha_\nu}$$

Kirchoff's Law

Means ~~when~~ emission & absorption are in balance $\left(\frac{dI_\nu}{ds} = 0\right)$

with a ~~the~~ source function = Planck Function.

Formal 1D Solution with Emission & Absorption

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

This is a linear 1st order diffy Q

Integrating factor is $e^{\int_0^{\tau_\nu} +1\, d\tau_\nu'} = e^{+\tau_\nu}$

$$e^{\tau_\nu}\frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = e^{\tau_\nu} S_\nu$$

NOTE: $\frac{d}{d\tau_\nu}\left[e^{\tau_\nu} I_\nu\right] = e^{\tau_\nu}\frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu$

Let's Define temporarily $\mathcal{J} \equiv e^{\tau_\nu} I_\nu$:

$$\int_{\mathcal{J}(0)}^{\mathcal{J}(\tau_\nu)} d\mathcal{J} = \int_0^{\tau_\nu} e^{\tau_\nu'} S_\nu(\tau_\nu')\, d\tau_\nu'$$

$$\mathcal{J}(\tau_\nu) = \mathcal{J}(0) + \int_0^{\tau_\nu} e^{\tau_\nu'} S_\nu(\tau_\nu')\, d\tau_\nu'$$

TOTAL OPTICAL DEPTH = CONST

(1) Substitute for $\mathcal{J} = e^{\tau_\nu} I_\nu$ and (2) Multiply equation by $e^{-\tau_\nu}$

$$\boxed{I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')} S_\nu(\tau_\nu')\, d\tau_\nu'}$$

Consider case of $S_\nu(\tau_\nu') = \text{CONST} = S_\nu$

$$I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu} + S_\nu \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')}\, d\tau_\nu'$$

Let $x = -(\tau_\nu - \tau_\nu') \Rightarrow dx = d\tau_\nu'$

[Remember: τ_ν is a CONST = TOTAL OPTICAL DEPTH!]

$$\int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')}\, d\tau_\nu' = \int_{-\tau_\nu}^{0} e^{x}\, dx = 1 - e^{-\tau_\nu}$$

$$\boxed{I_\nu = I_\nu(0)\, e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right)}$$