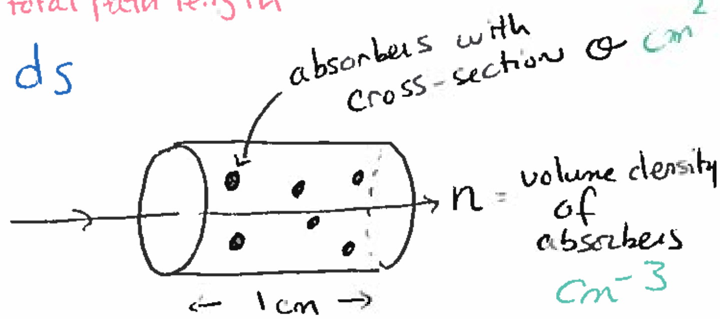
Optical Depth & Absorption Coefficient

ASTR 300B

$$\tau_\nu \equiv \int_0^L \alpha_\nu \, ds \quad \leftarrow L = \text{total path length}$$

$\alpha_\nu \approx$ # of photon absorptions per cm^{-1}



$$\underset{}{\alpha_\nu} = \underset{cm^{-3}}{n} \cdot \underset{cm^{2}}{\sigma_\nu} \qquad cm^{-1}$$

So really τ_ν is given by

$$\tau_\nu = \int_0^L n\,\sigma_\nu\, ds = \sigma_\nu \underbrace{\int_0^L n\, ds}_{\text{This is the Column Density}}$$

If σ_ν idependent of s

$$N \equiv \int \underset{cm^{-3}\cdot cm}{n\, ds} \qquad cm^{-2}$$

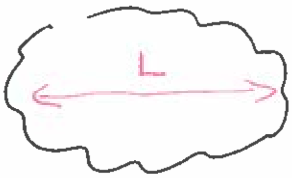
Column Density is a measure of how much "stuff" you have along a line of sight (in a particular direction).

$$\tau_\nu = \sigma_\nu \cdot N$$

Optical depth = cross-section of absorbers × Column Density of absorber

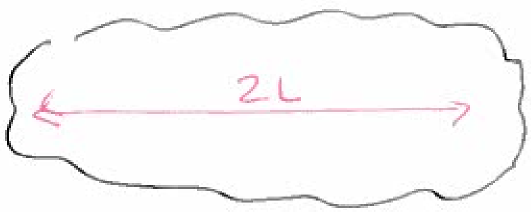
Practical Example :

Assume you have 2 clouds with absorbers with cross-sections σ_ν (same in both clouds) and n same in both clouds.



Cloud 1

$$N_1 = \int_0^L n\,ds = n \cdot L$$



Cloud 2

$$N_2 = \int_0^{2L} n\,ds = 2nL$$

$$\frac{\tau_\nu(\text{cloud 2})}{\tau_\nu(\text{cloud 1})} = \frac{2\cancel{\sigma_\nu}\cdot\cancel{n}\cdot\cancel{L}}{\cancel{\sigma_\nu}\cdot\cancel{n}\cdot\cancel{L}} = 2$$

Optical depth scales linearly with how much "stuff" you have along a line of sight.

Start with

$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$

$\frac{dI_\nu}{I_\nu} = -d\tau_\nu$

$I_\nu = I_\nu(0)\, e^{-\tau_\nu}$

Let's calculate expectation value of τ_ν

$$\langle \tau_\nu \rangle = \int_0^\infty \tau_\nu \cdot \underbrace{P(\tau_\nu)\, d\tau_\nu}$$

$\|$

Probability of absorption in $[\tau_\nu, \tau_\nu + d\tau_\nu]$

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Probability of non-absorption in $[0, \tau_\nu]$ and absorption within $d\tau_\nu$

Prob. photon absorbed $[0, \tau_\nu]$: $p(0, \tau_\nu) = \frac{\Delta I(\tau_\nu)}{I_0} = \frac{I_0 - I(\tau_\nu)}{I_0} = 1 - \frac{I(\tau_\nu)}{I_0}$

Prob. of non-absorption $[0, \tau_\nu]$: $1 - p(0, \tau_\nu) = \frac{I(\tau_\nu)}{I_0} = e^{-\tau_\nu}$ From soln. of 1D radiative transfer equation for pure absorption.

Prob. of absorption in $d\tau_\nu$: $p(\tau_\nu, \tau_\nu + d\tau_\nu) = \frac{dI(\tau_\nu)}{I(\tau_\nu)} = d\tau_\nu$

$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$

$\Rightarrow \left|\frac{dI_\nu}{I_\nu}\right| = |d\tau_\nu|$

Total Prob. is product of Prob of non-absorb $[0, \tau_\nu]$ $\times$ Prob. of absorb in $d\tau_\nu$

$$\langle \tau_\nu \rangle = \int_0^\infty \tau_\nu\, e^{-\tau_\nu}\, d\tau_\nu$$

NOTE: $\int x e^{-x} dx = -(1+x)e^{-x}$

$$= -\left(\cancel{\frac{1}{e^\infty} + \frac{\infty}{e^\infty}} \right) + (1+0)e^{-0}$$

$\lim \to 0$

$$= 1$$

when $\tau = 1$ $\quad \tau_\nu = \int_0^L \alpha_\nu\, ds = \alpha_\nu \cdot L$

$$\Rightarrow \quad 1 = \alpha_\nu \cdot L \quad \Rightarrow \quad L = \frac{1}{\alpha_\nu} = \frac{1}{n\sigma}$$

$\leftarrow$ Mean free path length