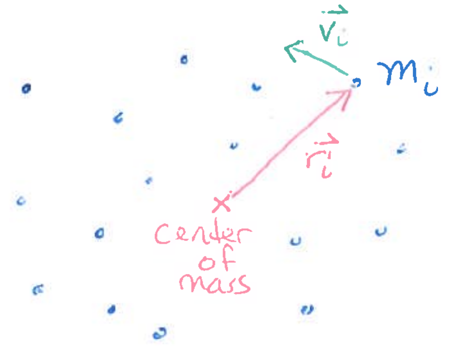
Virial Theorem Derivation

ASTR 25(

One of the most important dynamical theorems in astrophysics



Imagine a system of N <u>bound</u> particles with masses, m_i, a a distance/direction $\vec{r}_i$ from the center of mass.

$\vec{p}_i$ = momentum

Define the quantity A as :
$$A = \sum_{i=1}^{N} m_i \vec{v}_i \cdot \vec{r}_i$$

Now lets calculate $\frac{dA}{dt}$:

this is acceleration $\vec{a}_i$ — this is just $\vec{v}_i$

$$\frac{dA}{dt} = \sum_i m_i \frac{d\vec{v}_i}{dt} \cdot \vec{r}_i + m_i \vec{v}_i \cdot \frac{d\vec{r}_i}{dt}$$

$$= \sum_i \underline{m_i \vec{a}_i} \cdot \vec{r}_i + m_i \vec{v}_i \cdot \vec{v}_i$$

Note $\vec{v}_i \cdot \vec{v}_i = v_i^2$ scalar value

Newton's 2ND Law $\vec{F}_i = m_i \vec{a}_i$

$$= \sum_i \vec{F}_i \cdot \vec{r}_i + m_i v_i^2$$

this is $\frac{1}{2}$ of the kinetic energy $T = \sum_i \frac{1}{2} m_i v_i^2$

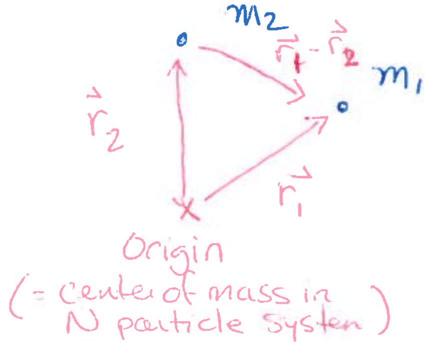
$$= \sum_i \vec{F}_i \cdot \vec{r}_i + 2T$$

Now if we take the time average $\left\langle \frac{dA}{dt} \right\rangle = \frac{1}{t}\int_0^t \frac{dA}{dt}\, dt = \left\langle \sum_i \vec{F}_i \vec{r}_i \right\rangle + 2\langle T$

Since the system of particles is <u>bound</u>, $\vec{r}_i$ and $m_i\vec{v}_i$ remain <u>finite</u>. This means A is <u>finite</u>. Therefore so is dA/dt. We say the system of particles is "Virialized" if after a long enough time $\left\langle \frac{dA}{dt} \right\rangle \to 0$

Virialized : $2\langle T \rangle = -\left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle$

Let's evaluate $\sum_i \vec{F}_i \cdot \vec{r}_i$ by considering the gravitational force between two masses:



unit vector in direction between i and j.

$$\vec{F}_i = \sum_{j \neq i} \frac{G m_i m_j \, \hat{r}_{ij}}{|\vec{r}_j - \vec{r}_i|^2} \qquad \hat{r}_{ij} = \frac{\vec{r}_j - \vec{r}_i}{|\vec{r}_j - \vec{r}_i|}$$

$$\vec{F}_i = \sum_{j \neq i} \frac{G m_i m_j (\vec{r}_j - \vec{r}_i)}{|\vec{r}_j - \vec{r}_i|^3}$$

Gravity For N Particles

For just 2 masses (m_1 and m_2) we have:

$$\vec{F}_1 \cdot \vec{r}_1 + \vec{F}_2 \cdot \vec{r}_2 = \frac{G m_1 m_2 (\vec{r}_2 - \vec{r}_1) \cdot \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} + \frac{G m_1 m_2 (\vec{r}_1 - \vec{r}_2) \cdot \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}$$

$$= \frac{G m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3} \cdot \left[(\vec{r}_2 - \vec{r}_1) \cdot \vec{r}_1 - (\vec{r}_2 - \vec{r}_1) \cdot \vec{r}_2 \right]$$

$$= \frac{G m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3} \cdot \left[(\vec{r}_2 - \vec{r}_1) \cdot (\vec{r}_1 - \vec{r}_2) \right]$$

$$= \frac{-G m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^{\cancel{3}}} \cdot \cancel{|\vec{r}_2 - \vec{r}_1|^2} = \frac{-G m_1 m_2}{|\vec{r}_2 - \vec{r}_1|} = + U_{12}$$

the gravitational potential !!

If we did this for 3 masses, we would get the sum of potential energies of the pairs (1,2) (2,3) and (3,1).

$$\left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle = \langle U \rangle$$

total potential energy of system

Virial Theorem: $\langle T \rangle = -\frac{1}{2} \langle U \rangle$

Valid when:
1. Bound system of particles
2. Force is purely radial ($F \sim \frac{1}{r^2}$ for gravity)
3. Virialized $\langle \frac{dA}{dt} \rangle \to 0$